LARGE OSCILLATIONS OF A PENDULUM: DEPENDENCE OF THE PERIOD ON THE SWING ANGLE

Activity Report No. 1.5

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Grading Scheme[1]

	ITEM	POINTS
Report Sections	**Title** The title should consist only of words that contribute directly to the report's subject.	
	Abstract The abstract should be a very succinct summary of the report. It should summarize the nature of the report, its rationale and the important findings. It should be written using passive tense and it should not exceed one page.	
	Introduction The introduction should offer immediate context for the reader by establishing why the problem being studied is important. Furthermore, it should illustrate the problem's nature and scope. The introduction should not exceed two pages.	
	Methods This section should include a description of what was done and what procedures were used to obtain the results. The description should be detailed enough to allow one to replicate the work being discussed. If the inclusion of extensive derivations is required for clarity, it may be placed in an appendix and simply referenced.	
	Results and Discussion This section contains a summary of the results along with their interpretation.	
	Conclusion[2] This section is for the presentation of the conclusions you have arrived at as they relate to your experiments or analysis.	
Report Format	**Paragraphing, Arrangement, Headings** Paragraphs are clearly focused, unified, and sensibly arranged. Transitions between and within paragraphs are thoughtfully chosen, and section headings, if included, are carefully worded and germane.	
	Formatting and Use of Figures, Tables and Equations Tables and figures are purposefully and professionally presented. Captions are well worded and the accompanying interpretation of tables and figures meaningful. Tables and figures work both independently and in context. All numbered figures and tables should be referred to in the text. The equations are treated as an integral part of the sentence and property punctuated.	
	Sentence structure The paper should exhibit ease with idiom. The writer should show facility with various sentence lengths and types so that the paper is highly readable. The paper should exhibit good use of grammar and syntax. Punctuation should enhance readability.	
	Spelling and Proofreading There are few or no spelling or typographical errors.	
	TOTAL	

[1] This table contains material taken from the book entitled "Style for Students: Effective Writing in Science and Engineering", by J. Schall, Burgess Publishing, Edina (MN), 1995.

[2] This section may be followed by one or more appendices. When an appendix is included, it should be properly typed and formatted in a way that is consistent with the rest of the report. However, it is possible to include hand written appendices. In this case, it is expected that they be legible and neat. When included, their contribution to the overall report grade will be lumped with that of the Conclusion section.

Abstract

The dependence of the period of oscillation of a simple pendulum, released from rest, on the initial angle is analyzed. Using a numerical approach, it is shown that the period of oscillation is indeed a strong function of the initial angle except when the angle is small.

1. Introduction

In Activity One, the period of the oscillations of a simple pendulum was experimentally measured to obtain the value of the acceleration of gravity g. The experiment in question was conducted by making a pendulum swing *gently*. Clearly, the concept of *gentleness* is hardly a precise and quantifiable scientific notion. In fact, in this context, a "gentle swing" actually means "*small* amplitude oscillations", where small is meant to indicate that the oscillation amplitude is within a range for which the oscillation period does not change significantly. The adverb significantly can be given a precise meaning by means of a convention, *e.g.*, by deciding that a significant change of the period of oscillation consists of 0.1 s (seconds).

From a practical viewpoint, the validity of the experimental method illustrated above is predicated upon the assumption that there exists a range of oscillation amplitudes for which the period is a constant. Hence, in order to justify the validity of the method and in order to gauge its accuracy, a study of the pendulum motion for any oscillation amplitude is necessary to determine whether or not it is true that there exists a range of oscillation amplitudes for which the period is independent of said amplitudes.

The overall objective of this activity is that of investigating the behavior of a simple pendulum, set in motion by changing the angle formed by the pendulum cord and the vertical direction, without imparting on the pendulum bob an initial velocity different from zero. More specifically, we want to obtain a curve representing the value of the period of oscillation as a function of the initial swing angle. This study will only include numerical experiments as the results presented will be obtained via computer simulations.

2. Methods

With reference to Fig. 1, consider a simple pendulum consisting of an inextensible cord of length L with a bob of mass m released from rest with an initial angle θ_0. In the calculations presented herein, L has been chosen to be equal to 1 m and the mass m has been chosen to be equal to 10 kg. The free-body and mass-acceleration diagrams for the system at hand are depicted in Fig. 2. By equating the FBD and the MAD[3] in Fig. 2, we derived the equations of motion for the pendulum in Fig. 1.

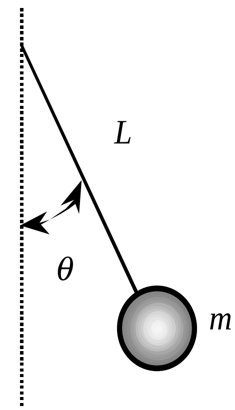

Figure 1: A simple pendulum.

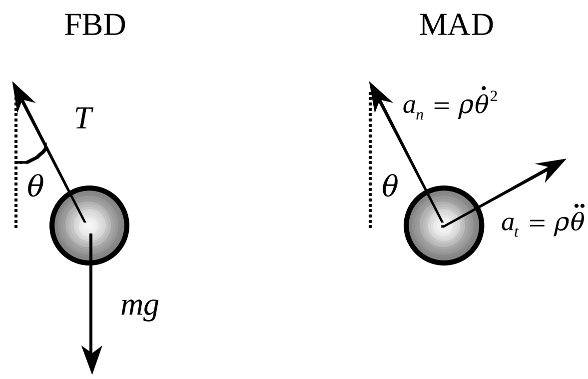

Figure 2: Free body and mass-acceleration diagrams for a simple pendulum.

In particular, the equation which describes the pendulum's free oscillations reads as follows:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0. \tag{1}$$

The above equation is rather difficult to solve analytically. Thus, we resorted to a numerical method to determine the equation's solution. In particular, we used the so-called modified Euler's method for the integration of ordinary differential equations. The numerical calculations have been carried out by coding the modified Euler's method in MATLAB (*c.f.* Appendix B for a listing of the MATLAB code). In order to study the dependence of the period on the initial swing angle, Eq. (1) was solved for a variety of angles, ranging from 0.01 rad to π/2 rad.

[3] The longhand calculations are reported in Appendix A.

3. Results and Discussion

Figure 3 depicts a series of solutions differing from one another by the value of $\theta(0)$, that is, the initial condition in terms of the swing angle θ, whereas the initial condition in terms of $\dot{\theta}$ is taken to be null for all cases.

Figure 3, displays the value of the swing angle as a function of time. The various curves differ in the initial swing angle.

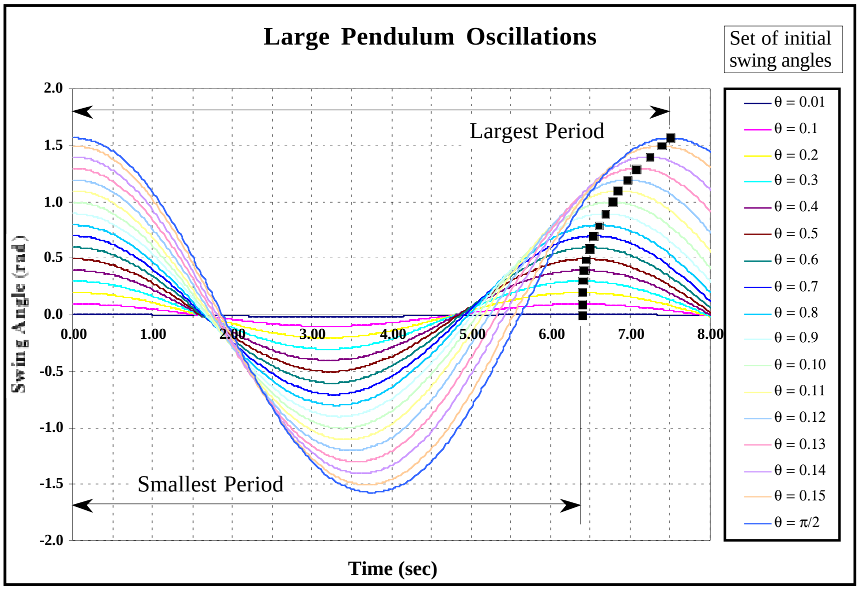

Figure 3: Plots of the swing angle as a function of time. The various curves differ in the value of the initial swing angle.

As it can be seen in the figure, the period of oscillation increases as the initial swing angle increases. Furthermore, it can be seen that the increase in period of oscillation becomes rather visible for values of the swing angles larger than 0.3 rad, whereas it is essentially negligible for angles smaller than 0.2 rad. In order to better illustrate this concept, consider the graph in Fig. 4. This graph depicts the value of the period of oscillation as a function of the initial swing angle. In addition to this function, the graph displays the actual value of the period of oscillation for the various data points. It should be noticed that the value of the period of oscillation for the first two data points is the same and coincides with the theoretical value that one would obtain by considering the linearized version of the ordinary differential equation which governs the small oscillations of a pendulum. It can also be noticed that once the value of the initial swing angle is larger than 0.35 rad the error in the estimate of the oscillation period is larger than 10%.

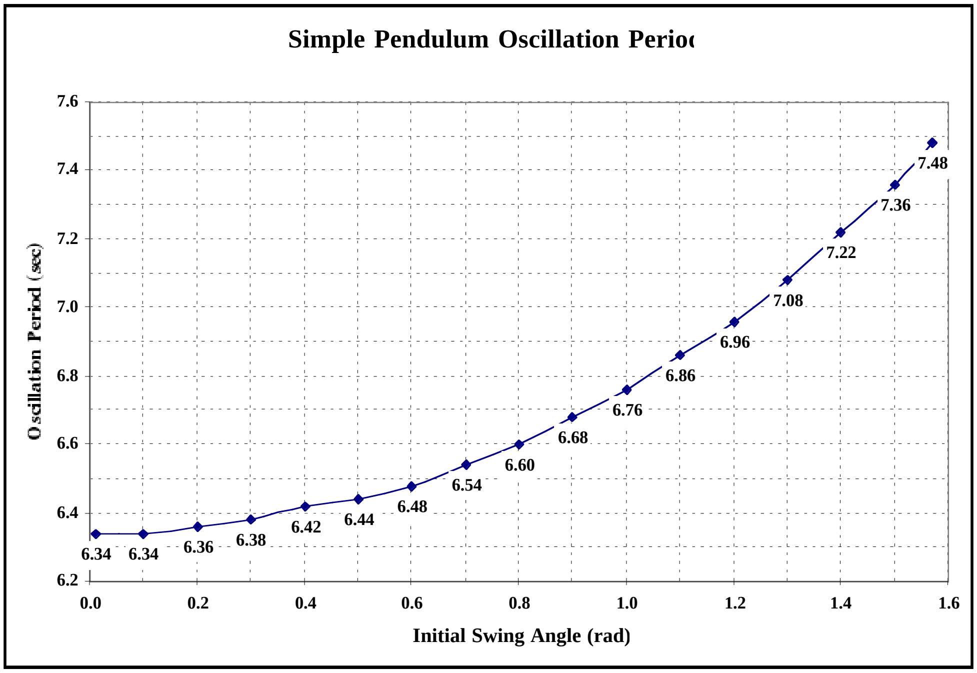

Figure 4: Plot of the period of oscillation as a function of the initial swing angle.

4. Conclusions

The data reported in the previous sections indicate that the period of oscillation of a simple pendulum that is released from rest can only be taken to be independent of the initial swing angle if the angle is sufficiently small. For the particular example reported herein, the expression *sufficiently small* was shown to correspond to angles less than 0.2 rad. This allows us to obtain estimates of the oscillation period within 10% error relative to the theoretical value obtained by considering the linearized equation of motion as opposed to the full nonlinear one.

As one may recall, in Activity One it was suggested that the initial swing angle not be taken to be greater than 6°, corresponding to roughly 0.1 rad. In view of the results shown in the previous section and with reference to Fig. 4 in particular, it can be said that such a recommendation allows one to compute estimates of the oscillation period with an error smaller than 1%. Such a small error margin is usually acceptable in engineering applications.

Appendix A

This appendix contains the long-hand derivations to obtain Eq. (1). With reference to Fig. 2, consider the sum of forces in the tangential direction, that is, in the direction perpendicular to the pendulum's cord. The only force that contributes to the resultant in this direction is the bob's weight mg, since the tension T acts in the normal direction. Also, notice that the component of the bob's weight in the tangential direction has its orientation opposite to the positive tangential direction determined by the orientation chosen for the angle θ. Hence we have

$$\sum F_t = -mg\sin\theta. \tag{2}$$

With reference to Fig. (3) and recalling that the pendulum cord is inextensible, we see that the acceleration in the tangential direction consists only of the term $\rho\ddot{\theta}$. Also, the path radius of curvature coincides with the length of the pendulum L. Therefore, equating the product of the acceleration and the mass in the tangent direction with the resultant of all the forces in that direction we obtain the following equation of motion (in the tangential direction):

$$-mg\sin\theta = mL\ddot{\theta}. \tag{3}$$

Dividing both sides of the equation above by mL and moving the right hand side to the left of of the equal sign, we obtain Eq. (1):

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0. \tag{4}$$

Equation (1) (or Eq. 4), is not the only equation of motion. In fact, one can repeat the process illustrated in this Appendix to derive the equation of motion in the normal direction. This would deliver an equation that allows one to compute the tension in the pendulum cord. However, this equation does not provide any useful information concerning the oscillation period and therefore will not be considered.

Appendix B

This appendix contains the listing of a MATLAB program for the integration of an ordinary differential equation of second of the form

$$\ddot{y} = f(x, y, \dot{y}), \tag{5}$$

using the modified Euler's method. A description of the modified Euler's method can found in the textbook used in class: "Engineering Mechanics: Dynamics", by Andrew Pytel and Jaan Kiusalaas, HarperCollins, New York, (1994).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [X,Y,YDot] = ModifiedEuler(xi,xf,y0,ydot0,N,fcn) % %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% ModifiedEuler() integrates a second order ODE of the form %
% y'' = fcn(x,y,y') whose name is contained in the string   %
% variable 'fcn'. The function named in the string variable %
% fcn must expect SCALAR input for x, y and y', and returns %
% a SCALAR.                                                 %
% The variables xi and xf define the extremes of the inter- %
% val of integration and must be such that xi < xf.         %
% The quantities y0 and ydot0 are the initial conditions.   %
% N is the number of integration steps.                     %
% The results are returned in the (row) vectors X, Y, and   %
% YDot, containing the coordinates of the of the evaluation %
% points and the corresponding values of y and y'.          %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
dx = (xf - xi) / N;
X(1) = xi; X(2) = X(1) + dx; X(3) = X(1) + 2 * dx;
YDot(1) = ydot0; Y(1) = y0;
YDot(2) = YDot(1) + dx * feval(fcn,X(1),Y(1),YDot(1));
Y(2) = Y(1) + dx * YDot(1);
YDot(3) = YDot(1) + 2 * dx * feval(fcn,X(2),Y(2),YDot(2));
Y(3) = Y(1) + 2 * dx * YDot(2);
for i = 4:N
      T(i) = T(i-2) + 2 * dx;
      YDot(i) = YDot(i-2) + 2 * dx * feval(fcn,X(i-1),Y(i-1),YDot(i-
1));
      Y(i) = Y(i-2) + 2 * dx * YDot(i-1);
end
X = X(:);
Y = Y(:);
YDot = YDot(:);
```