Uniaxial Loading: Design for Strength, Stiffness, and Stress Concentrations

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This overview of design concerning uniaxial loading is meant to supplement theoretical information presented in your text. It covers design for strength, stiffness, and stress concentrations. A detailed example is included.

Design for Strength

Strength is the most important component to safe design. Both normal and shear stresses must be considered.

Normal Stress

To determine dimensions for a safe design for normal stress in a uniform member, we must locate the place were the normal internal reaction is the greatest, perhaps by the method of sectioning or by drawing a load diagram. There are times when the area is not uniform, or dimensions change, but those scenarios will be covered under stress concentrations. For the present, we will consider a load P acting perpendicular to a constant cross-sectional area A which is to be determined. The stress, σ, is related as follows:

$$\sigma = \frac{P}{A} \Rightarrow A = \frac{P_{max}}{\sigma_{all}} \qquad (1)$$

P_{max} is the maximum internal force acting at the section of interest and σ_{all} is the allowable stress the material can sustain. For elastic materials, σ_{all} is usually determined by σ_{yield} /F.S. where F.S. is the Factor of Safety and σ_{yield} is the maximum stress a material can withstand without permanent deformation. σ_{yield} can be found for many materials in reference and/or textbooks. A factor of safety F.S. must be specified in all design projects; typical values for stress analysis are 1<F.S.<2.

Shear Stress

There is often a shear stress associated with an axially loaded member. It is commonly found in connections such as those made with bolts, pins, or glued or welded joints. The shear stress τ is related to the internal force V acting parallel to a constant cross sectional area A as follows:

$$\tau = \frac{V}{A} \Rightarrow A = \frac{V_{max}}{\tau_{all}} \qquad (2)$$

where V_{max} is the maximum internal shear force acting at the section of interest and τ_{all} is the allowable shear stress for the material of choice. τ_{all} is defined to be τ_{yield} /F.S. where τ_{yield} is the maximum shear stress a material can withstand without permanent deformation

and F.S. is the Factor of Safety in shear which may differ from that for normal stress, but usually does not.

If both normal and shear stresses act on the component being designed it is necessary to compute the areas from (1) and (2), compare them, and choose the larger of the two. Typically finding area A reduces to a single calculation such as length of a side or diameter of a rod. Next, round that number up (or down depending on the application) to the closest nominal size as dictated by commercial suppliers and standards organizations. Usual nominal dimensions in U.S. Customary units are given to the closest inch: 1, 1/8, 1/16, 1/32, etc. In S.I. units, they are usually given to the closest meter: centimeter, millimeter, etc.

Design for Stiffness

Stiffness, in the case of uniaxial loading, is associated with an allowable deformation: extension or contraction. This allowable value will either be provided in the problem statement, specified in a technical standard or code, or it may have to be deduced from the information provided. Under an axial load a member in tension lengthens, a member in compression shortens and deformation due to shear is usually not significant for design purposes. So here we need only be concerned with normal forces. The deformation is related to the internal normal load P, the length of the member L, the modulus of elasticity E, and the cross-sectional area A in the following way:

$$\delta = \frac{PL}{EA} \Rightarrow \frac{AE}{L} = \frac{P_{max}}{\delta_{all}} \Rightarrow \frac{A}{L} = \frac{P_{max}}{E\delta_{all}} \Rightarrow A = \frac{P_{max}L}{E\delta_{all}} \tag{3}$$

As one can see in (3), more information is needed with each successive equation. In design, this is helpful to us because it allows us to juggle multiple variables or to find more information by applying other design criteria.

You may encounter a member with several loads applied throughout the length, or one that has several different materials or cross sectional areas. The deformation of such a member is calculated by using the same formula but by applying vector addition to each section to obtain the total deformation.

$$\delta = \sum \frac{PL}{EA} \tag{4}$$

In this case the design is more open-ended.

Design for Stress Concentrations

Unlike the situation in the Design for Strength section, what happens if you encounter a non-uniform cross-section? If the nonuniformity is gradual, then the change in stress is smooth and a stress concentration may not occur. However, if the nonuniformity is abrupt then the stress changes sharply forming a stress concentration. The maximum stress σ_{max} created by the nonuniformity may often be determined by multiplying the

nominal stress σ_{nom}, which occurs in the same section, by a stress concentration factor K. In general the definitions are:

$$K = \frac{\sigma_{max}}{\sigma_{nom}} \text{ and } \sigma_{nom} = \frac{P}{A_{reduced}}$$
(5)

For common geometries K is tabulated in references. For instance, Pilkey (1997) provides an updated version of Peterson's (1974) graphical data and Roark and Young (1989) provides formulas. Uncommon geometries can be investigated using finite element analysis (an upper level numerical method). A key definition is A $_{reduced}$ which varies not only with geometry, but also between references, hence one must be careful to use it correctly. For a circular hole it is usually the width minus the diameter of the hole. Notably the thickness of a member containing a stress concentration does not enter into its value.

K values for a circular hole centered in an axially loaded plate can be calculated using the formula (Roark and Young (1989))

$$K = 3 - 3.13 * \frac{2r}{W} + 3.66 * \left(\frac{2r}{W}\right)^2 - 1.53 * \left(\frac{2r}{W}\right)^3$$
(6)

or found in graphs like the one below (simply a plot of the above formula)


K

2r/W
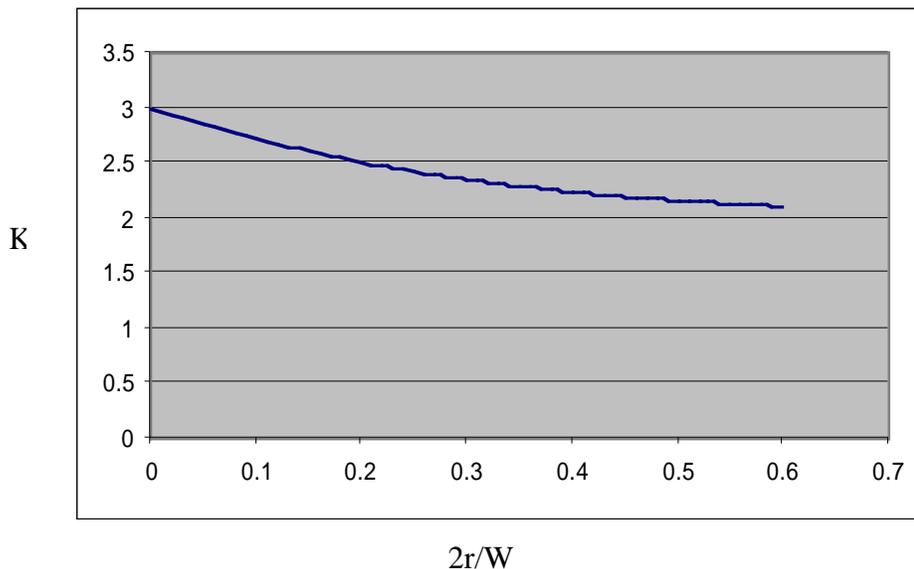
Here r is the radius of the hole and W is the width of the plate, not the thickness. It is interesting to note that the maximum K is 3.0, hence in the absence of data, one can design conservatively using this value.

For design, we set maximum stress σ_{max} equal to allowable stress σ_{all} and invert the stress concentration expression:

$$\sigma_{max} = K\sigma_{nom} = K\frac{P}{A_{reduced}} \Rightarrow A_{reduced} = K\frac{P}{\sigma_{all}} \qquad (7)$$

Since $A_{reduced}$ is necessary to find K which is yet unknown, we have a dilemma. The solution is to apply an iterative approach as shown in the design of the upper section of the bracket of example AD1.

Example AD1
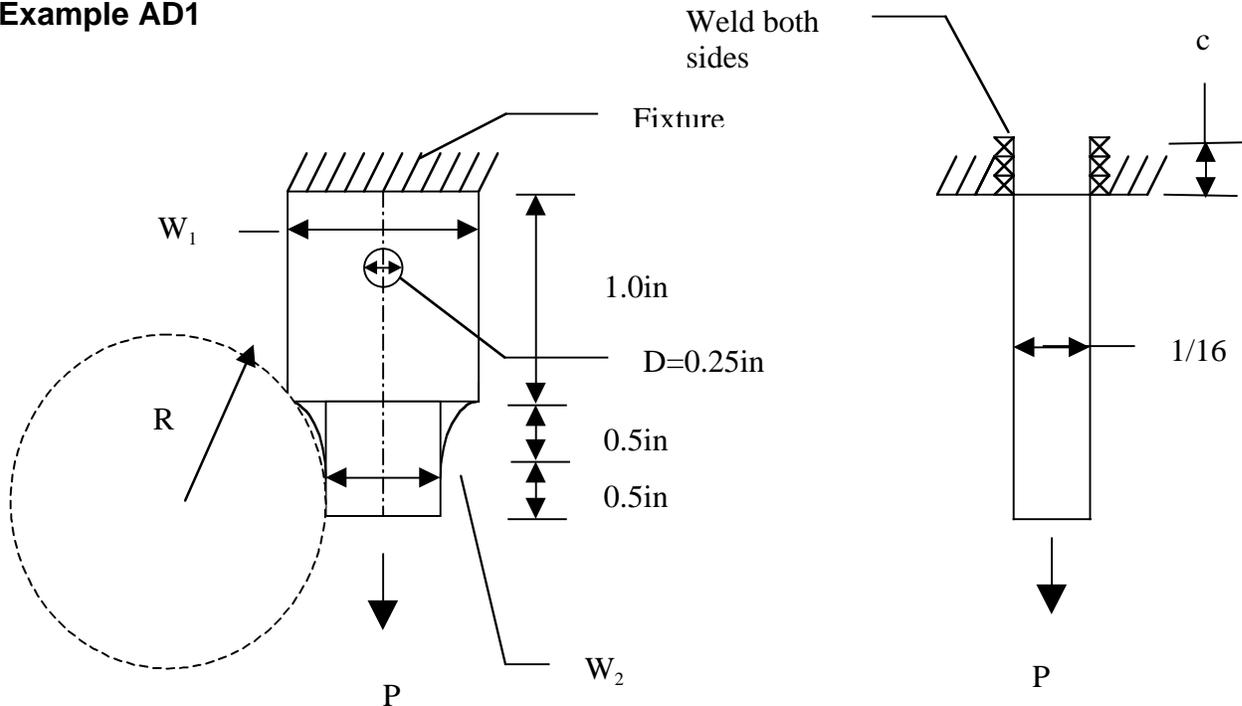
Design the A-36 steel hanging bracket so that it will carry a load of 1000 lb and not generate significant stresses in the transition from region of load application to location of the 1/4 inch diameter hole. The bracket has a thickness of 1/16 inch and is welded on both sides a depth c into the fixture. The bracket cannot deform while loaded more than 0.005 in. Use a factor of safety of 1.3.

Strength Design of Bracket

1. Make a list of everything you know about the problem.

You know:
- It has to hold a weight of 1000 lbs.
- That it has a 1/4 in. diameter hole in the top
- That it has a fillet in it.

- It is to be made of A-36 steel (σ_{yield} for A-36 steel is 36000 psi, E for A-36 steel is 29000 ksi (Hibbler (1997)).
- A factor of safety of 1.3 will be used.
- It has a thickness of 1/16 of an inch.
- The maximum deformation is 0.005 in.
- It is welded on both sides a depth c into fixture
- The length above the fillet is 1 in., the length where the fillet occurs is 0.5 in, and the length below the fillet is 0.5 in.
- The model for this problem is the given figure since it clearly shows the boundary conditions and the load.

2. What are we trying to find?

We are tying to find the dimensions of the bracket:
- the top width (W_1)
- the bottom width (W_2)
- the radius (R) of the fillet
- the depth (c) of the weld.

3. Think about how to go about starting the problem. A list of steps necessary to complete design of the bracket are:

- First, we will deal with the lower section of the bracket, neglecting the fillet.
- Second, we will move to the hole in the upper bracket.
- Third, we will deal with the fillet of the bracket.
- Fourth, we will determine the depth of the weld to prevent shear tearout.
- Last, we will check all of our dimensions by using the allowable deformation.

Lower Section

FBD:
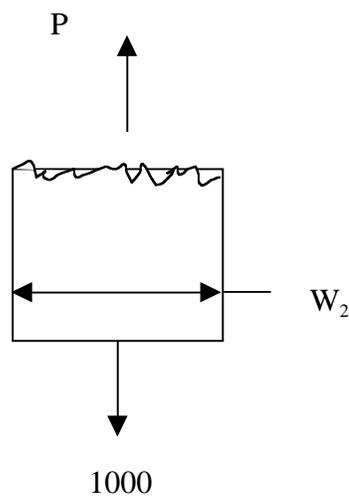
- First, perform a summation of forces to solve for P.
  $\Sigma F_Y : P - 1000 = 0 \Rightarrow P = 1000$

- Next consider the normal stress associated with the cut section.

$$\sigma = \frac{P}{A} \Rightarrow A = \frac{P_{max}}{\sigma_{all}} = \frac{P}{\sigma_{all}}$$

$$\sigma_{all} = \frac{\sigma_y}{F.S} = \frac{36000\,psi}{1.3} = 27692\,psi$$

$$A = \frac{1}{16} * W_2$$

$$W_2 = \frac{16P}{\sigma_{all}} = \frac{16 * 1000lb}{27692\,psi} = .5777in \approx 0.580in$$

Upper Section

This section needs more thought because we do not have all the necessary information to insert into an equation. We must:
- Guess at a value for the width of the upper section
- Substitute it into a formula or use a graph to find K, the stress concentration factor
- Calculate σ_{trial}
- Check σ_{trial} against an allowable stress to see if our guess was accurate within a reasonable tolerance or not.

FBD: Stress Distribution:

P

D=0.25 in

W₁

1000

σ_{max}

$\sigma_{far\ from}$
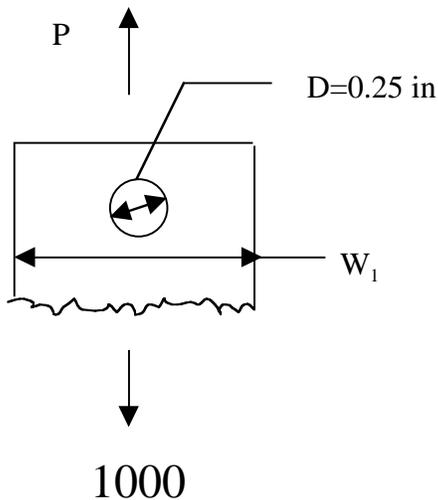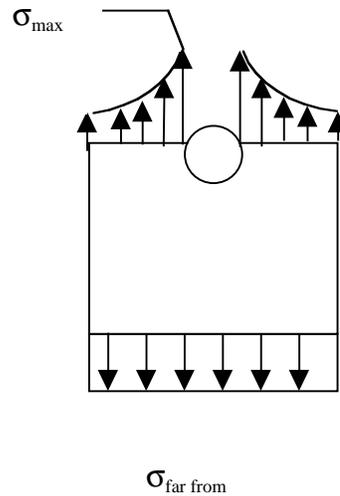
NOTE:
1. In this case, we must design for $\sigma_{max} = \sigma_{all}$
2. σ_{nom} applies to the reduced cross sectional area (i.e., the width minus the diameter of the hole)

$$\sigma_{max} = K\sigma_{nom} = K\frac{P}{A_{reduced}} = K\frac{P}{t(W_1 - D)}$$

$$\sigma_{all} = K\frac{16P}{W_1 - 0.25}$$

- We must first choose a W_1 to start with and find the K value corresponding to it. The following formula is found in Roark and Young (1989); it defines the value of K for a hole based on geometric properties.

$$K = 3 - 3.13 * \frac{2r}{W} + 3.66 * \left(\frac{2r}{W}\right)^2 - 1.53 * \left(\frac{2r}{W}\right)^3$$

Lets start with W_1=1 inch. For W_1=1 inch, K=2.422. Also check graph of K values.

For this case,

$$\sigma_{trial} = K\frac{16P}{W_1 - 0.25} = 2.422 * \frac{16 * 1000}{1 - 0.25} = 51700\, psi$$

$\sigma_{trial} > \sigma_{all}$ so our guess of 1 inch was too low; choose a greater W_1.

- Our next try will be two inches. For W_1=2 inches, K= 2.663

$$\sigma_{trial} = K\frac{16P}{W_1 - 0.25} = 2.663 * \frac{16 * 1000}{2 - 0.25} = 24320\, psi$$

This time, $\sigma_{trial} < \sigma_{all}$ by a significant amount, which means if we use this value we will be wasting material. We want to find a value between one and two inches that will be perfect for our needs. We can graph these two values to help us visualize where we want our value of w to
be (Note: the actual graph will not be linear).
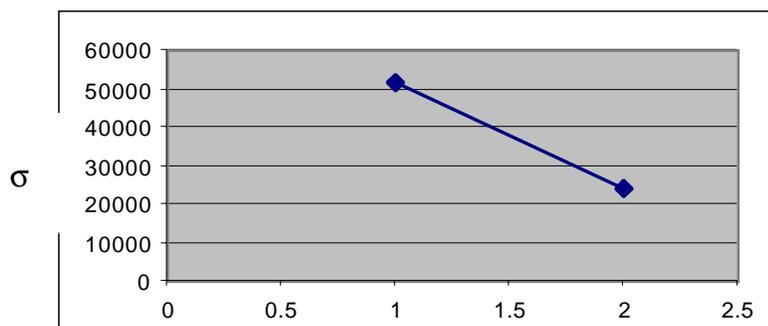
From the graph, you can see that 1.75<W_1<2 inches is a good place to hit our target range of 27692 psi. We shall perform a linear interpolation in order to find a satisfactory value for W_1.

$$\frac{W_{desired} - W_{1in}}{W_{2in} - W_{1in}} = \frac{\sigma_{all} - \sigma_{1in}}{\sigma_{2in} - \sigma_{1in}} \Rightarrow \frac{W_{desired} - 1}{2 - 1} = \frac{27692 - 51700}{24320 - 51700} \Rightarrow W_{desired} = 1.877in$$

For W=1.88, K=2.644
$$\sigma_{trial} = K\frac{16P}{W_1 - 0.25} = 2.644 * \frac{16 * 1000}{1.8768 - 0.25} = 26000\,psi \le 27700\,psi \Rightarrow 6\% \text{ within } \sigma_{yield}$$

where we round σ_{yield} to 27700 psi. So if we accept σ_{max} within 6.1% of σ_{all}, then choose W_1=1.875= 1 7/8 in a good nominal dimension. But lets check W_1=1.75in:

For W_1=1.75, K= 2.623
$$\sigma_{trial} = K\frac{16P}{W_1 - 0.25} = 2.623 * \frac{16 * 1000}{1.75 - 0.25} = 27978 \ge 27692$$

1.75 and 1.88 give us values between our desired value, so we will try to average them with W_1=1.815in

W_1=1.815in, K=2.634
$$\sigma_{trial} = K\frac{16P}{W_1 - 0.25} = 2.634 * \frac{16 * 1000}{1.815 - 0.25} = 26932\,psi$$

W_1=1.815in is the closest value we have come upon and seems to be in the right range for our work. When we round 1.815 up to the closest nominal size we get 1 7/8 in.

Fillet

Here we will compute a value for the radius of the fillet. For a fillet, the K value depends on three geometric properties: W_1, W_2, and R, the radius of the fillet as seen is Hibbeler (1997) figure 4-23. We have found two out of the three problem requirements. However,

$$\frac{w}{h} = \frac{W_1}{W_2} = \frac{1.815}{.577} = 3.129 \text{ and for that line in the graph, as } \frac{r}{h} \text{ goes to one, K}\approx1.3. \text{ So we}$$

will set our value for r greater than h. But we must check to make sure that our fillet does not overcome the allowable stress. So, for our check:

$$\sigma_{all} \ge K\sigma_{avg} = \frac{K * P}{W_2 * (\frac{1}{16})} = \frac{1.3 * 1000}{0.5777 * (0.0625)} = 35862.06$$

$$\sigma_{all} = 27692.31 \le 35862.06$$

So our value for W_2 is unacceptable and we must choose another one considering the stress concentration at the fillet.

$$\sigma_{all} = K\frac{P}{A} = 1.3 * \frac{1000}{W_2 * 0.0625} = 27692 \Rightarrow W_2 = 0.7511in.$$
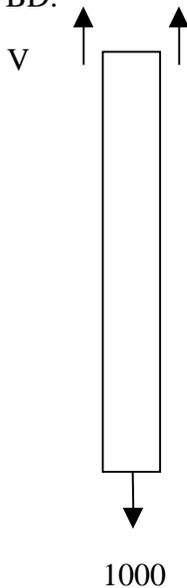
For a nominal value we will choose 7/8 in or 0.875in

For our new value of W_2 our value of K will be slightly affected, but will still have a value of approximately 1.3, it will be slightly lower, but that will only ensure our safe design.

Shear Stress

Now we must turn our attention to the shear stress associated with the weld of the bracket. We must determine the dimension c so that we do not experience shear tear out. Our first step is to determine the shear yield stress for A-36 steel. That information is not provided in your text so we must make an assumption.

Assume that A-36 steel behaves like aluminum for which the data is given. For 2014-T6 aluminum, σ_{yield} in compression and tension is 60 ksi. τ_{yield} is 25 ksi (Hibbeler 1997). This is approximately 42% of the yield stress for compression/tension. For 6061-T6 aluminum σ_{yield} is 37 ksi in compression and tension and τ_{yield} is 19 ksi. This is approximately 52% of the strength for compression and tension. In assuming that A-36 steel behaves like aluminum we will pick the smaller of these two values, namely 42%, and use that percentage as our shear strength for A-36 steel. In doing so we use 42% of 36 ksi, or 15.1 ksi. Applying a factor of safety, we find that $\tau_{all} = 11.61$ ksi. To find the depth, c, we will apply the shear stress formulas.

FBD:

V

1000

First, we will do summation of forces:
$$\sum F_y : 1000 - 2V = 0 \Rightarrow V = 500$$

$$\tau = \frac{V}{A} \Rightarrow A = \frac{V}{\tau_{all}} \Rightarrow W_1 * C = \frac{V}{\tau_{all}} \Rightarrow C = \frac{V}{W_1 * \tau_{all}} \Rightarrow C = \frac{500}{1.815 * 11610} = 0.0237 \approx 0.025in$$

For a nominal value we will choose 1/32 or 0.03125.

Deformation

To determine the deformation of the bracket, we will break it into three sections and perform vector addition to each section to determine whether or not our dimensions are large enough to prevent an unacceptable deformation. We will break it up as follows: the section above the fillet , the average of the section of the fillet, and the section below the fillet.

- Start with the section above the fillet and apply formulas. We will use the reduced area for the area.

$$\delta = \frac{PL}{EA} = \frac{1000lb * 1in}{29,000,000 * (0.0625 * (1.815 - 0.25))} = 0.00035in.$$

- Next, find the average area of the fillet section by approximating with a straight line.

$$A = W * T = \left(\frac{1.75 + 0.7511}{2}\right) * T = 0.080,$$ then use the area found to find the deformation of that section.

$$\delta = \frac{PL}{EA} = \frac{1000 * 0.5}{29000000 * 0.08} = 0.00022in$$

1. Third, the area below the fillet.

$$\delta = \frac{PL}{EA} = \frac{1000 * 0.5}{29000000 * (0.0625 * 0.7511)} = 0.00036$$

4. Last, add all the sections together.

$$\delta = \sum \frac{PL}{EA} = 0.00035 + 0.00022 + 0.00036 = 0.00093$$ which is well below our allowed value of 0.005 in. So, for this problem, our dimensions satisfy the stiffness requirement.

Decision:

For our final dimensions we have:
- W_1=1.875in
- W_2=0.8125=13/16 in
- R=1in
- C=1/32in

References

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Peterson, Rudolph Earl (1974). Stress Concentration factors, charts and relations useful in making strength calculations for machine parts and structural elements. John Wiley & sons, New York.

Pilkey, Walter D. (1997). Peterson's stress concentration factors. John Wiley & sons, New York.

Roark, Raymond J. and W.C. Young (1989). Formulas for Stress and Strain, 5th ed., McGraw-Hill Book Co., New York.