Column Buckling: Design using Euler Theory

Our use of Euler's buckling formula here is NOT intended for actual design work, but ONLY as a vehicle to illustrate design concepts and process which will carry over to a more sophisticated approach. With this limitation in mind, the treatment here supplements sections on Euler buckling of columns given in your textbook. More practical formulas are given in Hibbeler (1997), sections 13.6, 13.7. We cover design for buckling, physical insight on feasible designs and the concept of a design space and limitations of the method. We close with an example and problems.

Design for Buckling

Buckling is of concern whenever a slender member is under compression and in this case we refer to it as a column problem even if it is neither vertical nor an architectural column. If its physical length L is known and the compressive load has been set, then the design for buckling of a column reduces to three issues: (1) settle on the end or boundary conditions and determine its effective length L_{eff} (KL in Hibbeler, 1997), (2) determine the material, second moment of area product EI that is sufficient to prevent buckling and (3) insure that this cross section has sufficient area A so that the compressive stress is less than the allowable stress (also known).

The easiest concept to grasp is that the design load P_{des} must be less than the critical buckling load P_{cr} which, of course, is given by a formula. For our purposes this formula is Euler's and we write

$$P_{des} < P_{cr} \equiv \pi^2 EI / L_{eff}^2 \qquad (1)$$

but we can easily replace the rightmost formula by another more accurate one. It is important to realize that the effective length of a column is that which deflects into the shape of a half sine wave and that P_{cr} is not an actual load, but a load-independent number which is a characteristic of the geometry and material of the column. (Mathematically it is associated with the eigenvalue of the system.) Moreover, since the design load is fixed, yet must be less than this number, it is this number that must be increased by appropriate choice of material and geometry. But by how much? This is set by the safety factor for buckling.

The safety factor FS for buckling (usually greater than other safety factors) dictates the ratio between P_{des} and P_{cr}. So building upon (1),

$$P_{des} \le \frac{P_{cr}}{FS} = \frac{\pi^2 EI}{FS\ L_{eff}^2} \Rightarrow EI \ge (FS\ P_{des} L_{eff}^2)/\pi^2 \qquad (2)$$

The expression on the right is the design equation where everything on its right-hand side is known or set for the time being. FS is generally prescribed by a code or specification. Therefore the left-hand side can be solved for either E or I, that is, either material or cross section geometry. Usually the material is already known for other reasons and we use (2) to solve for I.

For example, if the material is known and a round cross section is desired, $I = \pi(r_0^4 - r_i^4)/4$. Then we can solve for either r_0 or r_i according to the following:

$$r_0^4 \geq r_i^4 + \frac{4(FS)P_{des}L_{eff}^2}{\pi^3 E} \qquad r_i^4 \leq r_0^4 - \frac{4(FS)P_{des}L_{eff}^2}{\pi^3 E} \tag{3}$$

But we are not quite done.

We must insure that our geometry has large enough cross section area so that

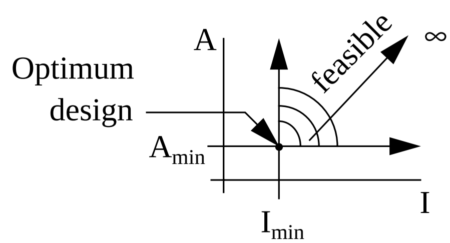
$$P_{des}/A \leq \sigma_{all} \tag{4}$$

Now we are done.

Feasible Designs and Design Space

Physically, what are we doing to design for buckling? Clearly feasible designs feature both sufficiently large *I* to resist bending and *A* to decrease compressive stress. The load is fixed a priori so that even stress is adjusted not by changing load, but rather by adjusting *A* in P_{des}/A. Furthermore, we can change the boundary conditions to adjust the effective length. If the situation defies our normal approach given by the equations above, we can radically adjust effective length by introducing braces, e.g., divide the column into two or more shorter lengths. We have completely turned the analysis problem on its head.

The solution for geometric variables, like *I* and *A*, lead to the concept of design space. Simply put, we are searching for a nominal or stocked ‘column’ to satisfy load and stress requirements. Many solutions exist. We seek the optimum. This means we may have to iterate or search a few times to get a good design (see the figure). Whereas today we may search catalogs, tomorrow in advanced work we can learn to build mathematical search engines to optimize the design.



Limitations

Instability of columns involves three cases: global buckling of the column (it buckles as a unit) which is covered above; local buckling which is localized failure of a compression region in a column (e.g., waviness in a web or local kinking in a tube wall); and torsional or twisting instability in thin-walled members related to shear flow (shear flow but not torsional instability is covered in Hibbeler § 7.4, 7.5). CAUTION: Global buckling predicted by Euler’s formula severely over estimates the response and under estimates designs. The latter two modes of buckling are covered in advanced courses.

Example

BuD1. Design a round lightweight push rod, 12 in long and pinned at its ends, to carry 500 lb. The factors of safety are 1.2 for material and 2.0 for buckling.

Solution:

1. For minimum weight, choose 6061-T6 aluminum bar. Properties from Hibbeler (1997): $E = 10 \times 10^6$ psi, $\sigma_Y = 37$ ksi, $\tau_Y = 19$ ksi. Divide yield stresses by factor of safety FS to get allowables $\Rightarrow \sigma_{all} = 37/1.2 = 30.83$ ksi, $\tau_{all} = 19/1.2 = 15.83$ ksi.
2. $I \geq (FS\ P_{des} L_{eff}^2)/(E\pi^2) = 2.0(500)(12^2)/(10^7\pi^2) = 0.001459\ \text{in}^4$
3. $\pi d^4/32 = I \geq 0.001459\ \text{in}^4 \Rightarrow d \geq 0.349\ \text{in}$
4. Search of nominal sizes yields: Choose 3/8 in diameter bar.
5. $P_{des}/A = 500/(\pi d^2/4) = 4530\ \text{psi} \ll 30,800\ \text{psi} = \sigma_{all}$ Check O.K.
6. Final check: $P_{cr} = \pi^2 EI/L_{eff}^2 = \pi^2(10^7)(\pi \cdot 0.375^4/32)/12^2 = 1330\ \text{lb}$
   $\therefore P_{cr}/FS = 1330/2 = 665\ \text{lb} > 500\ \text{lb} \equiv P_{des}$ Check O.K.
7. The extreme overdesign indicated by the check in step 5 means that a round bar is not the best cross sectional shape. Perhaps a tube or pipe would be optimal. Why? Nonetheless, step 6 yields an overdesign of (665 - 500)/500 $\rightarrow$ 33% which is more reasonable, but still high because the next larger stock bar size was 3/8 and 23/64 in (0.359 in) is not stocked. Decision: Choose 3/8 in dia. Al 6061-T6 bar.

Problems

BuD2. Redo problem BuD1 to seek a more optimal design, i.e., reduce the cross sectional area so that the stress check is closer to the allowable. Explain why your design works (or does not!).

References

Hibbeler, R.C. (1997) Mechanics of Materials, 3rd ed., Prentice Hall, Englewood Cliffs.