Bending: Design for Strength, Stiffness and Stress Concentrations

This overview of the theory for design for bending of beams is meant to supplement that given in your textbook. It is based upon the Bernoulli Euler theory which is applicable to most common engineering applications. First we cover design for strength because for safety reasons structural integrity is engineering's first priority. Second we cover design for stiffness, but doing so in this order does not mean that strength always controls the design (see the example). After commenting on stress concentrations and limitations at this level of knowledge, we close with an example and problems.

Design for Strength

Once the loading scenarios are determined, we can draw shear and moment load diagrams for a beam under each scenario. For example, if we know the load for a beam, but its location is uncertain, then for one scenario we would locate the load to achieve a maximum moment; for the other we would locate it to achieve maximum shear. Let's say a simply-supported beam is to carry a concentrated force P, then for scenario 1 locate P at midspan and for scenario 2, locate P just inside either support. Further, if the cross section is rectangular of area A, then the maximum shear stress equals P/A (by simple shear!) for Scenario 2 rather than 3P/4A (by flexural shear) for Scenario 1. Can you show both of these results? And by doing so, thereby convince yourself that consideration of critical load scenarios is important.

In beam design the normal stress obtained from maximum moment M_{max} usually dominates over shear stress obtained from maximum shear V_{max} (but for exceptions, see limitations below). Therefore, our first choice to obtain the section geometry is to use

$$\sigma = \frac{My}{I} = \frac{M}{I/y} \Rightarrow S = \frac{I}{c} = \frac{M_{max}}{\sigma_{all}} \tag{1}$$

where S is the so-called section modulus, c is the maximum value of y and the allowable stress would be known. For example, if the cross section is rectangular of dimensions b $\times$ h so that I = $bh^3/12$ and c = h/2, then $S = bh^2/6$. Hence knowing either b or h, we can use (1) to get the other. Next for this geometry, we must check the shear stress using either or both of the following:

$$\begin{aligned} \tau_{max} &= (V_{max}Q_{max})/(It_{min}) \text{ , if the shear is due to flexure} \\ \tau_{max} &= V_{max}/A \text{ , if the shear is not due to flexure (simple shear case)} \end{aligned} \tag{2}$$

and then demonstrate that $\tau_{max} \le \tau_{all}$ where the allowable shear stress would be known. If this check fails, then one must determine the geometry from (2) and check that $\sigma_{max} \le \sigma_{all}$. Otherwise we could compute geometry for each case and compare dimensions to reach a decision, but this approach provides a less evident argument and should be a second choice.

A change in cross section geometry along the beam presents a further consideration. For this, see the section on stress concentrations in bending below.

Design for Stiffness

To design for stiffness, an allowable kinematic condition must be specified. For beam bending, this would normally be a bound upon deflection v like $v_{max} \le v_{all}$. Although less likely, a maximum slope could be specified instead. Once a model with appropriate boundary conditions has been established, the maximum deflection or slope would be obtained using either double integration of the moment equation or a table for the solution. The relevant equations in sequence are

$$EI\frac{d^2v}{dx^2} = M(x) \Rightarrow EIv_{max} = \text{value} \Rightarrow I \ge \frac{\text{value}}{Ev_{max}} \tag{3}$$

where value is obtained by substituting the location of maximum deflection x into the deflection equation obtained from the differential equation in (3). This is illustrated in the example below.

Stress Concentration Equations

Stress concentrations in bending arise when uniformity of geometry is disrupted. In this case, a design problem may become iterative if initially we do not know the beam geometry required to determine the stress concentration factor. Then the design algorithm is: 1) solve the problem for dimensions of a uniform prismatic beam, 2) determine the stress concentration factor for the disruption in the uniform beam and use it to check if allowables are violated, and 3) repeat until optimal dimensions which satisfy all requirements are found.

Hibbeler (1997) provides stress concentration factors for symmetric fillets and grooves in the top and bottom of beams. Factors for geometries including holes and both one-sided and circumferential grooves under bending, torsion and axial deformation are given using Neuber's diagram by Ugural and Fenster (1979). But a more general and very reliable source is Roark and Young (1975). When applying formulas from any source, it is important to note the definition of the nominal or reference stress and the stress concentration factor itself.

Limitations

Although Bernoulli-Euler theory is very good, its application above is for elementary design of straight beams; we neither considered curved beams (Hibbeler, 1997, § 6.8) nor accounted for:

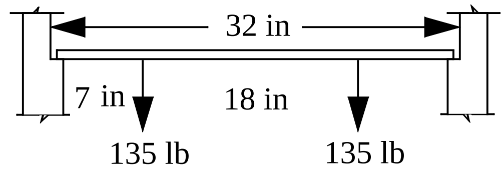
1. Situations in which the shear stress in the beam is the same order of magnitude as the normal stress. This occurs if large shear loads are applied or the beam is short. Principle stresses, obtained using transformation equations or Mohr's circle (see Hibbeler, § 11.2), become important, particularly in concentrations at flange-to-web junctions. Short beams do not flex much and their deflection is due mainly to shear, not moment. A beam is said to be short if its length is less than 10 times its section depth[1]. This situation is covered in advanced courses.
2. Local buckling and rotational instability of beams which involves three cases: global buckling of the structure (it buckles as a unit) which is covered subsequently in these notes; local buckling which is localized failure of a compression region in a beam (e.g., waviness in a web or local kinking); and torsional or twisting instability in thin-walled members related to shear flow (Hibbeler, § 7.4, 75).

Nonetheless, this methodology will carry the day in many situations, but the engineer must always be aware of hazardous or special situations and alert all involved parties.

1. Definition of a short beam is due to Zhuravskii, 1821-1891, and is discussed in Gere and Timoshenko (1997).

Example

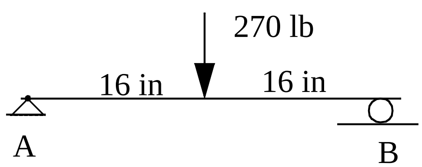
BD1. Design a round chinning bar to fit between a jamb 32 in wide and support a 270 lb person. Client specifications are: (1) minimize weight, (2) set grip spread to 18 in as shown in the figure, (3) diameter of bar to be about 1 in, (4) minimize deflection and limit it to 1/2 in and (5) use a factor of safety FS = 1.2.



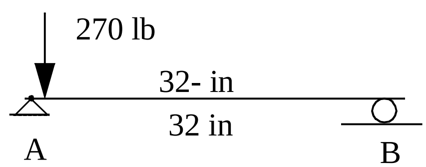
Solution:

1. For minimum weight, spec #1, choose 6061-T6 aluminum tube or pipe. Properties from Hibbeler (1997): E = 10×10^6 psi, σ_Y = 37 ksi, τ_Y = 19 ksi. Divide yield stresses by factor of safety FS to get allowables $\Rightarrow$ σ_{all} = 37/1.2 = 30.83 ksi, τ_{all} = 19/1.2 = 15.83 ksi.
   (NOTE: Ryerson (1987-89) gives σ_Y = 37 ksi for Alclad Al 6061-T6[1] and σ_Y = 40 ksi for Al 6061-T6. Our use of the lower value for this material is conservative and therefore safe.)
2. Prescribe the loading scenarios. The most critical load will occur if the person hangs from the bar with one hand. This is an obvious possibility and will cover the client's spec #2. Ranked in order from the most critical (Students should show this!), the loading scenarios are:
   Scenario 1: Central concentrated load. Maximum moment occurs if the entire load is placed in the center of the bar. This will also generate a maximum deflection.
   Scenario 2: One-sided concentrated load. Significant (but unknown) simple shear occurs if the load is placed very near to either support, say the left.
3. Prescribe the models for each scenario.

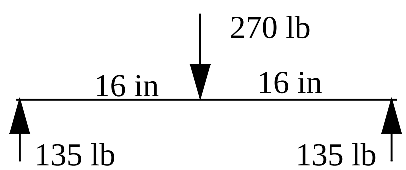
Scenario 1:



Scenario 2:



4. Freebody diagram and statics for Scenario 1
   Clearly $F_A = F_B = V_{max}$ = 135 lb
   $M_{max\,@\,\mathrm{center}}$ = (135)(16) = 2160 in lb
   (Note: Students should draw V-M diagrams to verify these "max" values.)



5. Strength design for Scenario 1:
   Client spec #3 $\Rightarrow$ Set r_0 = 1.0/2 = 0.5 in
   and solve for r_i where these are the outer and inner radii, respectively. Using Eq. 1,

$$\sigma_{all} = \frac{M_{max} r_o}{\pi(r_0^4 - r_i^4)/4} \Rightarrow r_i = \left[r_0^4 - \frac{4M_{max} r_0}{\pi\sigma_{all}}\right]^{1/4} = 0.365 \text{ in} \Rightarrow t_{wall} = r_0 - r_i = 0.134 \text{ in}$$

   Check catalog (Ryerson, 1987-89) for aluminum tube with nominal size nearest to 1" OD $\times$ .134" wall. There the largest wall thickness for 1" tube is 0.125. Too small. Looking further the

1. Alclad denotes a special aluminum alloy coating on both sides of a standard aluminum substrate (see Ryerson).

next size tube with commensurate wall size is 1 1/2 in OD which greatly exceeds spec #3. However, 3/4 in pipe has dimensions: 1.05" OD × 0.113" wall. Try this. Substituting r_0 = 1.05/2 = 0.525 in into the above formula and solve for r_i and wall thickness; we get:

$$\sigma_{all} = \frac{M_{max} r_o}{\pi(r_0^4 - r_i^4)/4} \Rightarrow r_i = \left[r_0^4 - \frac{4M_{max} r_0}{\pi\sigma_{all}}\right]^{1/4} = 0.413 \text{ in} \Rightarrow t_{wall} = r_0 - r_i = 0.112 \text{ in}$$

Excellent! Choose 6061-T6 aluminum pipe: 3/4 in O.D. × 0.113 in wall (0.824 in I.D.).

6. Comments on Step 5:

   a) For this choice of pipe: $\sigma_{max} = M_{max} r_0 / I = 30.6 \text{ ksi} < 30.8 \text{ ksi} \equiv \sigma_{all} \Rightarrow$ Check O.K.

   Here we used radii for the pipe so that $I = \pi(r_0^4 - r_i^4)/4 = 0.037 \text{ in}^4$.

   b) Warning: Ryerson (1987-89) does NOT indicate a heat treatment (T-value like T6) which is important to yield strength. Heat treatment should be settled upon prior to purchase.

   c) Interestingly, adequately thick tubing is standard stock in steel, but not in aluminum.

7. Check deflection; client spec. #4:

$$v_{max} = \frac{PL^3}{48EI} = \frac{270(32)^3}{48(10\times10^6)(0.037)} = 0.498 \text{ in} < 0.5 \text{ in} \Rightarrow \text{ Spec. \#4 satisfied.}$$

   where the formula for v_{max} comes from double integration of the differential equation in (3) for a simply supported beam under a concentrated load P applied at midspan, i.e.,

$$\iint EI\frac{d^2v}{dx^2} = \iint \frac{P}{2}x = -\frac{Px}{48}(3L^2 - 4x^2), \left(0 \le x \le \frac{L}{2}\right) \Rightarrow EIv_{max} = \frac{PL^3}{48} \text{ @ } x = \frac{L}{2}$$

   NOTE: Alternatively we could have used the rightmost equation in (3) to arrive at the tube or pipe dimensions and in turn used the leftmost equation in (1) as a check on strength.

8. Check the flexural shear stress:

   For a solid semi-circular section, $Q = (\pi r^2/2)(4r/3\pi) = 2r^2/3$. Then

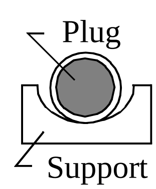
$$Q = Q_{whole} - Q_{hole} = \frac{2}{3}(r_0^3 - r_i^3) = 0.0498 \text{ in}^3 \Rightarrow \tau_{max} = \frac{V_{max} Q}{It} = 173 \text{ psi} < \tau_{all} \Rightarrow \text{O.K.}$$

   where τ_{all} = 15.8 ksi. Students should verify this equation, especially the result for Q.

9. Scenario 2. Only a strength check for simple shear is necessary. The deflection and reaction at B are negligible (Can you show this?), hence a freebody diagram is obvious (see illustration of Scenario 2 above). $\tau_{max} = V/A = 270/\pi(1.05^2 - 0.824^2)/4 = 2550 \text{ psi} < \tau_{all} \Rightarrow$ O.K.

10. Decision. Choose 6061-T6 aluminum pipe: 3/4 in O.D. × 0.113 in wall. The design is tight, both max stress and deflection are more than 99% of the allowables, but the margin of safety may be increased by using 6061-T6 rather than Alclad 6061-T6 (see note in step 1).

    Warnings: (1) Make certain the T6 heat treatment is applied to the aluminum.
    (2) Both ends of the pipe should be plugged with stiff plastic or metal to prevent local buckling of the pipe wall by distributing concentrated support reactions throughout the pipe wall at each end of the bar (see the figure).



Problems

BD2. Redo problem BD1, but use 1018 cold drawn steel tube which has a minimum yield strength of 65 ksi. Compute the final weight of your design and compare it with that for Example BD1.

BD2. Design a scaffold plank to span 10 ft between simple supports such that it is 12 in wide (11 1/4 in dressed) softwood and supports three 200 lb men who each occupy 18 in length of beam and together cause no more than 1/16 in deflection. Given: σ_{all} in bending is 900 psi, $E = 1.0 \times 10^6$ psi and τ_{all} parallel to the grain is 250 psi.

References

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