Adhesion between elastic cylinders based on the double-Hertz model

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A cohesive zone model for two-dimensional adhesive contact between elastic cylinders is developed by extending the double-Hertz model of Greenwood and Johnson (1998). In this model, the adhesive force within the cohesive zone is described by the difference between two Hertzian pressure distributions of different contact widths. Closed-form analytical solutions are obtained for the interfacial traction, deformation field and the equilibrium relation among applied load, contact half-width and the size of cohesive zone. Based on these results, a complete transition between the JKR and the Hertz type contact models is captured by defining a dimensionless transition parameter \( \mu \), which governs the range of applicability of different models. The proposed model and the corresponding analytical results can serve as an alternative cohesive zone solution to the two-dimensional adhesive cylindrical contact.

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1. Introduction

Adhesive forces that act between contacting bodies play a key role in determining the mechanical behavior of small-scale systems. For instance, adhesive force can induce significant local stress in atomic force microscopy (AFM) which can therefore result in substantial wear and tip degradation (Liu et al., 2010). With increasing usage of micro-scale components and devices, it is imperative to obtain a better understanding of the contact behavior considering adhesive forces.

Since Hertz’s seminal work (1882) on the unilateral contact of elastic spheres, numerous studies have been conducted on the adherence of spherical bodies. Bradley (1932) examined the attractive force between two rigid spheres by considering the molecular interactions. Later on, two famous models for adhesive contact between elastic spheres were proposed by Johnson et al. (1971) (JKR model) and Derjaguin et al. (1975) (DMT model), respectively. However, the magnitudes of the pull-off force predicted by the JKR and DMT models are quite different. Tabor (1977) then compared the two models and showed that JKR and DMT models represent two limiting cases of adhesive contact and their ranges of validity can be assessed by a dimensionless parameter (i.e., Tabor parameter) (Greenwood, 1997; Johnson and Greenwood, 1997; Barthel, 2008). To be more specific, the JKR model works well for soft materials with relatively high surface energy while the DMT model is more appropriate for hard solids with low surface energy. The first cohesive zone model which can allow for the transition between the JKR and DMT models was established by Maugis (1992). In this model (the so-called Maugis–Dugdale (M–D) model), the adhesive stress acting over the cohesive zone is assumed to be constant (i.e., Dugdale (1960)), which facilitates the derivation of analytical solutions. Soon afterwards, this model was also extended to describe the noncontact case (Kim et al., 1998).

In parallel with the M–D model, Greenwood and Johnson (1998) put forward an alternative cohesive zone model, known as the double-Hertz (D–H) model, which is also applicable to arbitrary values of Tabor parameter. In this model, the adhesive force within the cohesive zone is described by the difference between two Hertzian pressure distributions of different contact radii. It was found that results obtained by the D–H model are very close to those from the M–D model. However, the D–H model is more analytically tractable than the M–D model since the corresponding analysis relies solely on the classical Hertzian solutions. For this reason, the D–H model is often adopted to study the adhesion behavior of complex contact systems involving rough contact surfaces (Persson, 2002; Zhang et al., 2014), viscoelastic materials (Haiat et al., 2003) and functionally graded elastic solids (Jin et al., 2013). Recently, the D–H model was reconsidered in a slightly different context using an auxiliary function method (Barthel, 2012).

The above advances in contact mechanics of three-dimensional spherical bodies laid a solid foundation for the study of two-dimensional cylindrical contact systems. Barquins (1988) developed the...
JKR-type solutions for elastic cylinders and verified it experimentally. With use of Barquins’s theory, Chaudhury et al. (1996) predicted the surface and adhesion energies of elastomeric polydimethylsiloxane (PDMS) successfully. The two-dimensional JKR model was also extended to the non-slipping case with the frictionless contact assumption relaxed (Chen and Gao, 2006a; 2007) and the conforming contact case with the half-plane assumption relaxed (Sundaram et al., 2012).

The above mentioned JKR-based models, however, do not consider the adhesion forces outside the contact area and therefore are only applicable to soft bodies with relatively large Tabor parameters. For general material properties, Baney and Hui (1997) proposed the first cohesive zone model for cylindrical contact in the framework of M–D model, Morrow and Lovell (2005) then extended Baney and Hui’s theory to the case where the surfaces are not within intimate contact but are within the range of adhesive interaction. The same two-dimensional M–D analysis was also performed by Johnson and Greenwood (2008) independently, with emphasis on the pull-off force. Chen and Gao (2006b) presented an analogous M–D model of a cylinder in non-slipping adhesive contact with a stretched substrate. Furthermore, based on the two-dimensional M–D model, Sari et al. (2005) also investigated the sliding and rolling motion of a cylinder on the substrate subjected to combined normal and tangential forces.

The present study is aimed to extend the three-dimensional double-Hertz model of Greenwood and Johnson (1998) to a plane strain problem, with emphasis on establishing a set of simple analytical solutions which are applicable for a full range of Tabor parameters. These solutions can not only describe a complete transition between the two-dimensional JKR and the Hertz type contact models, but also exhibit as equally effective as the two-dimensional M–D model.

The rest of the paper is organized as follows. We first extend the double-Hertz model to the cylindrical contact system in Section 2. The main analytical results are then presented in dimensionless form in Section 3. Section 4 discusses the reduction of the proposed model in two limiting cases of small and large cohesive zones. The traction-separation relation within the cohesive zone is examined in Section 5. Finally, some concluding remarks are provided in Section 6.

2. Two-dimensional double-Hertz model

Fig. 1a shows the adhesive contact between two dissimilar elastic cylinders with parallel axes under a prescribed load $P$ (with unit N/m and negative when tensile). Contact occurs over a rectangular region of width $2a$. In fact, if the tangential tractions are neglected, this problem is equivalent to the plain strain frictionless contact problem between a rigid cylinder of radius $R$ and an elastic half-plane with a effective Young’s modulus $E^*$, where

$$1/R = 1/R_1 + 1/R_2$$  \hspace{1cm} (2.1)

and

$$1/E^* = (1 - v_1^2)/E_1 + (1 - v_2^2)/E_2,$$  \hspace{1cm} (2.2)

respectively. In Eqs. (2.1) and (2.2), $R_1$, $R_2$ are the radii, $v_1$, $v_2$ are the Poisson ratios and $E_1$, $E_2$ are the Young’s moduli of the contacting cylinders, respectively (Johnson, 1985).

For subsequent analytical treatment, as shown in Fig. 1b, a Cartesian coordinate system $(x, z)$ is set up with origin at the center of the contact zone and $z$ direction pointing into the half-plane. The distribution of surface traction consists of two terms: the Hertz pressure $p_H$ acting on a contact region of width $2a$ and the adhesive tension $p_A$ acting on an interaction zone of width $2c$. The noncontact regions bounded by half-widths $a$ and $c$ (i.e., $a < |x| < c, z = 0$) are known as the cohesive zones. Since the present problem is symmetry with respect to the $z$-axis, we only quote the equations for $x \geq 0$ in the following analysis.

In the absence of adhesive force, the Hertz-type pressure distribution between a rigid cylinder and an elastic half-plane is given by (Johnson, 1985)

$$p(x) = \frac{E^*}{2R} (a^2 - x^2)^{1/2}, \hspace{1cm} |x| \leq a$$  \hspace{1cm} (2.3)

which corresponds to a prescribed load

$$P = \frac{\pi a^2 E^*}{4R}$$  \hspace{1cm} (2.4)

The derivative of the surface normal displacement with respect to $x$ can be expressed as

$$\frac{\partial u_z}{\partial x} = \frac{x}{R}, \hspace{1cm} 0 \leq x \leq a,$$  \hspace{1cm} (2.5a)

$$\frac{\partial u_z}{\partial x} = -\frac{2}{\pi E^*} \int_0^a \frac{p(s)}{x-s} \, ds = \frac{x - \sqrt{x^2 - a^2}}{R}, \hspace{1cm} x \geq a.$$  \hspace{1cm} (2.5b)

According to Greenwood and Johnson (1998), the essential idea behind the proposed two-dimensional double-Hertz model is to represent the adhesive tensile traction by resorting to the difference of two Hertzian pressure distributions, that is,
\[ p(x) = \frac{E}{2R} \left[ (c^2 - x^2)^{1/2} - (a^2 - x^2)^{1/2} \right], \quad 0 \leq x \leq a, \tag{2.6a} \]
\[ p(x) = \frac{E}{2R} (c^2 - x^2)^{1/2}, \quad a \leq x \leq c. \tag{2.6b} \]

Furthermore, we also have
\[ \frac{\partial u_x}{\partial x} = 0, \quad 0 \leq x \leq a \tag{2.7a} \]
and
\[ \frac{\partial u_x}{\partial x} = -\frac{x}{R} + \frac{x - \sqrt{x^2 - a^2}}{R} = -\frac{\sqrt{x^2 - a^2}}{R}, \quad a \leq x \leq c, \tag{2.7b} \]
respectively.

Denoting \( p_0 = \alpha E/2R \), Fig. 2 plots the distributions of the normalized pressures \( p/p_0 \) resulting from the difference between two Hertzian solutions with contact half-widths \( a \) and \( c \) as shown in Eq. (2.6). It can be observed from this figure that the ellipsoidal pressure distribution over \( a \leq |x| \leq c \) steadily decreases from the maximum value at \( x = a \) to zero at \( x = c \). In the following, the pressure in Eq. (2.6b) scaled by an arbitrary factor \( \lambda (> 0) \) will be employed to model the adhesive tensile traction over \( a \leq |x| \leq c \), resulting in the final distribution of surface traction when combined with an original unscaled Hertzian pressure. Under this treatment, the interfacial traction for \( 0 \leq x \leq c \) can be written as
\[ p_a(x) = -\frac{\lambda E}{2R} \left[ (c^2 - x^2)^{1/2} - (a^2 - x^2)^{1/2} \right], \quad 0 \leq x \leq a, \tag{2.8a} \]
\[ p_a(x) = -\frac{\lambda E}{2R} (c^2 - x^2)^{1/2}, \quad a \leq x \leq c. \tag{2.8b} \]

Denoting the maximum magnitude of interfacial traction as
\[ \sigma_0 = \frac{\lambda E}{2R} (c^2 - a^2)^{1/2}, \tag{2.9} \]
which can be chosen, somewhat arbitrarily, to match that of the Lennard–Jones potential. Eq. (2.8) can be re-written in terms of \( \sigma_0 \) as
\[ p_a(x) = -\frac{\sigma_0}{(c^2 - a^2)^{1/2}} \left[ (c^2 - x^2)^{1/2} - (a^2 - x^2)^{1/2} \right], \quad 0 \leq x \leq a, \tag{2.10a} \]
\[ p_a(x) = -\frac{\sigma_0}{(c^2 - a^2)^{1/2}} (c^2 - x^2)^{1/2}, \quad a \leq x \leq c. \tag{2.10b} \]

which corresponds to an applied load
\[ P = \frac{\pi E}{4R} \left[ a^2 - \lambda (c^2 - a^2) \right]. \tag{2.11} \]

Within the cohesive zone \( a \leq x \leq c \), the derivative of the surface normal displacement is given by
\[ \frac{\partial u_x}{\partial x} = \frac{(1 + \lambda) \sqrt{x^2 - a^2}}{R} \frac{x}{R} \tag{2.12} \]
and the resulting separation between the rigid cylinder and the deformed half-plane surface is obtained from the geometric relation as
\[ h = -\delta + \frac{x^2}{2R} + u_x, \tag{2.13} \]
and accordingly
\[ \frac{dh}{dx} = \frac{1 + \lambda}{2R} \sqrt{x^2 - a^2}, \quad a \leq x \leq c. \tag{2.14} \]

In Eq. (2.13), \( \delta \) denotes the indentation depth at contact center. Recalling the fact that \( h(a) = 0 \), the separation can be derived from Eq. (2.14) as
\[ h(x) = \frac{1 + \lambda}{2R} \left[ x \sqrt{x^2 - a^2} - a^2 \ln \left( \frac{x + \sqrt{x^2 - a^2}}{a} \right) \right], \quad a \leq x \leq c. \tag{2.15} \]

The surface energy is defined as the work needed to separate two surfaces to infinity. Since the separation vanishes for \( 0 \leq x < a \) and the traction vanishes for \( x \geq c \), we have
\[ \Delta \gamma = -\int_0^\infty p_a(h) dh = -\int_a^c p_a(x) \frac{dh}{dx} dx = \frac{\lambda (1 + \lambda) E}{2R} \int_a^c \sqrt{x^2 - a^2} (c^2 - x^2) dx = \frac{\lambda (1 + \lambda) E}{6R^2} \left[ c (c^2 + a^2) K \left( \frac{\sqrt{c^2 - a^2}}{c} \right) - 2ca^2 \Pi \left( \frac{\sqrt{c^2 - a^2}}{c} \right) \right], \tag{2.16} \]
where \( K(\cdot) \) and \( E(\cdot) \) are the complete elliptic integral of the first and second kinds, respectively.

To determine \( \lambda \), a transition parameter is introduced as (Baney and Hui, 1997)
\[ \mu = \frac{4}{\pi^2 \eta^2} \mu_T = 4 \left( \frac{R \Delta \gamma^2}{\pi^2 E^2 z_0^2} \right)^{1/3} \approx 4 \sigma_0 \left( \frac{R}{\pi^2 E^2 \Delta \gamma} \right)^{1/3}, \tag{2.17} \]
where \( \mu_T \) denotes the classical Tabor parameter, which represents the ratio of the elastic displacement of the surfaces at pull-off to the effective range of surface forces characterized by \( z_0 \) (Tabor, 1977). Under this condition, a relationship between \( \lambda \) and \( \mu \) can be established by combining Eqs. (2.9) and (2.17) as follows
\[ \mu = 2\lambda (c^2 - a^2)^{1/2} \left( \frac{E}{\pi^2 R^2 \Delta \gamma} \right)^{1/3}. \tag{2.18} \]

Up to this point, main equilibrium equations of the two-dimensional double-Hertz model have been established.

3. Nondimensional results

In this section, the above results are summarized in a dimensionless form. By introducing the following nondimensional parameters:
\[ a^* = \frac{a}{\eta}, \quad c^* = \frac{c}{\eta}, \quad m = \frac{c}{a}, \quad P^* = \frac{P}{(\pi RE^2 \Delta \gamma)^{1/3}} \tag{3.1} \]
with
\[ \eta = 2 \left( \frac{R^2 \Delta \gamma}{\pi E} \right)^{1/3}, \]  
(3.2)
the dimensionless normal load can be obtained from Eq. (2.11) as
\[ P^* = a^2 - \lambda (c^2 - a^2), \]  
(3.3)
and the relationship between \( a \) and \( c \) defined in Eq. (2.16) can be normalized as
\[ 1 = \frac{4 \lambda (1 + \lambda)}{3 \pi} \left[ c'(c^2 + a^2)E \left( \frac{\sqrt{c^2 - a^2}}{c} \right) - 2c' a^2 K \left( \frac{\sqrt{c^2 - a^2}}{c} \right) \right], \]  
(3.4)
where \( \lambda \) is related to the transition parameter through
\[ \mu = \frac{4 \lambda}{\pi} (c^2 - a^2)^{1/2}. \]  
(3.5)
By eliminating \( \lambda \), the \( c \sim a \) relation can also be established in an implicit form as
\[ 3(c^2 - a^2) = \mu \left( \sqrt{c^2 - a^2} + \frac{\pi}{4} \right) \times \left[ c'(c^2 + a^2)E \left( \frac{\sqrt{c^2 - a^2}}{c} \right) - 2c' a^2 K \left( \frac{\sqrt{c^2 - a^2}}{c} \right) \right]. \]  
(3.6)

Eqs. (3.3)–(3.5) can be compared to the two-dimensional Hertzian (Johnson, 1985), JKR (Barquins, 1988) and Maugis–Dugdale (Baney and Hui, 1997; Johnson and Greenwood, 2008) results, which are
\[ P_{\text{Hertz}} = a^2, \quad P_{\text{JKR}} = a^2 - 2 \sqrt{a^*}, \]  
(3.7a, b)
and
\[ P_{\text{M–D}} = a^2 - \mu a \sqrt{m^2 - 1} \]  
(3.7c)
with
\[ 1 = \frac{\mu}{2} a^2 \left[ m \sqrt{m^2 - 1} - \ln(m + \sqrt{m^2 - 1}) \right] + \frac{\mu^2}{2} a^2 \left[ m \sqrt{m^2 - 1} - \ln(m + \sqrt{m^2 - 1}) - m \ln m \right]. \]  
(3.7d)

Fig. 3 displays the equilibrium \( P-a \) curves in the present double-Hertz (D-H) model under various values of \( \mu \) according to Eqs. (3.3)–(3.5). The corresponding Hertz, JKR and M–D solutions are also included for comparison. It can be seen that the D-H curves nearly coincide with the M–D solution and they approach the Hertz solution as \( \mu \) is reduced to zero (e.g., \( \mu = 0.01 \)). This is different from the adhesion problem of spheres, where the DMT solution is recovered for \( \mu \to 0 \). For moderate values of \( \mu \) (e.g., \( \mu = 4 \)), some deviation from the JKR solution remains noticeable especially for small contact sizes. For sufficiently large \( \mu \) (e.g., \( \mu = 10 \)), the JKR curve is readily approached. In fact, the JKR curve is expected to be fully recovered in the limit of \( \mu \to \infty \), which will be discussed in detail in the next section.

Fig. 4 plots the half-widths of contact and interaction zones versus the applied load for various values of \( \mu \). From this figure, for larger \( \mu \) the contact size \( a \) and the interaction zone size \( c \) are nearly the same, whilst for lower \( \mu \), \( c \) is far more than \( a \). As the cohesive zone size is bounded by \( a \) and \( c \), higher \( \mu \) corresponds to a smaller cohesive zone whereas lower \( \mu \) results in a larger cohesive zone.

In addition, the interfacial traction can be normalized as
\[ p^*(x') = \begin{cases} a' \sqrt{1 - x'^2} - \frac{\eta}{4 \sqrt{m^2 - 1}} (\sqrt{m^2 - x'^2} - \sqrt{1 - x'^2}), & |x'| \leq 1 \\ -\frac{a' \sqrt{\pi^2 \gamma}}{4 \sqrt{m^2 - 1}}, & 1 < |x'| < m \end{cases} \]  
(3.8)
where
\[ p^* = \frac{p}{\zeta}, \quad x' = \frac{x}{a}, \quad \zeta = \left( \frac{\eta a^2 \Delta \gamma}{\pi R} \right)^{1/3}. \]  
(3.9)

The interfacial traction distributions for different values of \( \mu \) for the case of \( P^* = 1 \) are illustrated in Fig. 5. It can be observed that the interfacial traction varies from a compressive value at contact center, through a maximum tensile value at contact edge, to zero at interaction fringe.

For comparison, Fig. 6 plots the D-H interfacial traction distributions for \( P^* = 1 \) and \( \mu = 4 \), along with the corresponding Hertz, JKR and M–D curves according to
\[ p_{\text{Hertz}}(x') = a' \sqrt{1 - x'^2}, \]  
\[ p_{\text{JKR}}(x') = a' \sqrt{1 - x'^2} - a' \frac{1}{2} (1 - x'^2)^{1/2}, \quad |x'| \leq 1, \]  
(3.10a, b)
and
\[ p_{\text{M–D}}(x') = \begin{cases} a' \sqrt{1 - x'^2} - \frac{\eta}{4} \arctan \sqrt{\frac{x'^2}{1 - x'^2}}, & |x'| \leq 1, \\ -\frac{a' \sqrt{\pi^2 \gamma}}{4 \sqrt{m^2 - 1}}, & 1 < |x'| < m. \end{cases} \]  
(3.10c)
the applied load in Eq. (3.3) and the \( c \sim a \) relation in Eq. (3.6) become
\[
P' \cong a^2 - \frac{\mu \pi a^2}{4} \sqrt{2e},
\]
(4.2)
\[
1 \cong \frac{\pi \mu^2 a^2}{24e} \left[ (2 + 2e + e^2)E\left(\sqrt{\frac{2e + e^2}{1 + e}}\right) - 2K\left(\sqrt{\frac{2e + e^2}{1 + e}}\right) \right],
\]
(4.3)
respectively.

Recalling the following asymptotic expansions for \( |z| < 1 \):
\[
K(z) = \frac{\pi}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 z^2 + \left( \frac{13}{24} \right)^2 z^4 + \cdots + \left( \frac{(2n-1)!!}{(2n)!!} \right)^2 \frac{z^{2n}}{2n-1} + \cdots \right],
\]
(4.4a)
\[
E(z) = \frac{\pi}{2} \left[ 1 - \left( \frac{1}{2} \right)^2 \frac{z^2}{1} - \left( \frac{13}{24} \right)^2 \frac{z^4}{3} - \cdots - \left( \frac{(2n-1)!!}{(2n)!!} \right)^2 \frac{z^{2n}}{2n-1} - \cdots \right].
\]
(4.4b)

Eq. (4.3) reduces to
\[
\epsilon \cong - \frac{32}{\pi^2 \mu a^4}.
\]
(4.5)

Inserting Eq. (4.5) back into Eq. (4.2) yields
\[
P' \cong a^2 - 2 \frac{\sqrt{\frac{\pi^2 \mu^2}{4}}}{a},
\]
(4.6)
which is precisely the two-dimensional JKR result (Barquins, 1988).

Note that Eq. (4.5) implies an additional condition on the contact half-width \( a \) as the JKR limit is recovered, i.e.,
\[
\epsilon \cong - \frac{32}{\pi^2 \mu a^4} = \frac{4E\Delta \gamma}{\pi \sigma_0^2} \ll 1.
\]
(4.7)

It is clear that the condition in Eq. (4.7) is generally valid as \( \mu \to \infty \), and hence the JKR result can be viewed as the limit case of the present double-Hertz model for \( \mu \to \infty \). For any finite value of \( \mu \), however, too small \( a' \) cannot ensure Eq. (4.7) to be satisfied. This accounts for the noticeable deviation from the JKR curve at small \( \epsilon' \) for moderate value of \( \mu (\mu = 4) \), as shown in Fig. 3.

4.2. Large cohesive zone

When the cohesive zone is large compared to the contact zone, i.e., \( m \gg 1 \), the applied load in Eq. (3.3) and the \( c \sim a \) relation in Eq. (3.6) are reduced to
\[
P' \cong a^2 - \frac{\pi \mu}{4} c',
\]
(4.8)
\[
1 \cong \frac{\mu}{3} c' \left( c' + \frac{\pi}{4} \mu \right).
\]
(4.9)
Combining Eqs. (4.8) and (4.9) yields
\[
c' = - \frac{\pi \mu}{8} \left( 1 - \sqrt{1 + \frac{192}{\pi^2 \mu^2}} \right).
\]
(4.10)
\[
P' \cong a^2 + \frac{\pi^2 \mu^2}{32} \left( 1 - \sqrt{1 + \frac{192}{\pi^2 \mu^2}} \right).
\]
(4.11)
For \( \mu \ll 1 \), Eq. (4.10) is reduced to
\[
c' \cong \frac{\sqrt{3}}{\mu},
\]
(4.12)
and accordingly
\[
P' \cong a^2,
\]
(4.13)
5. Traction-separation relation

For various ratios of the interaction zone size \( c \) to contact zone size \( a \), the dependence of the adhesive traction on the surface separation within the cohesive zone is implicitly determined by combining Eqs. (2.10b) and (2.15). In fact, it is confined between two asymptotic limits: small cohesive zone and noncontact cohesive zone.

5.1. Small cohesive zone

As the cohesive zone becomes very small (\( c \to a \)), the surface separation given by Eq. (2.15) is reduced to

\[
h(x) \equiv h_c \left( \frac{x^2 - a^2}{c^2 - a^2} \right)^{2/3}, \quad a \leq x \leq c.
\]

where

\[
h_c = \frac{1 + \lambda}{12Ra} \left( c^2 - a^2 \right)^{1/3}.
\]

Recalling the adhesive traction given in Eq. (2.10b), the traction-separation relation can be obtained as

\[
\frac{p_A}{\sigma_0} = \left[ 1 - \left( \frac{h}{h_c} \right)^{2/3} \right]^{1/2}.
\]

Accordingly, the surface energy is

\[
\Delta \gamma = \int_0^{h_c} p_A dh = \frac{3 \pi}{16} \sigma_0 h_c.
\]

In contrast, the corresponding relation in the M–D model is

\[
\Delta \gamma = \sigma_0 h_m,
\]

where \( h_m \) is the corresponding separation at \( x = \pm c \) and \( \sigma_0 \) is a uniform adhesive stress acting over the entire cohesive zone.

5.2. Noncontact cohesive zone

At the instant of detachment, the contact half-width shrinks to zero (\( a \to 0 \)) for the first time, the adhesive traction, the surface separation and the surface energy behave as

\[
p_A(x) = -\sigma_0 \left( 1 - \frac{x^2}{c^2} \right)^{1/2}, \quad 0 \leq x \leq c,
\]

\[
h(x) = h(c) \frac{x^2}{c^2}, \quad 0 \leq x \leq c,
\]

\[
\Delta \gamma = \lambda (1 + \lambda) \frac{Ec^3}{6R}.
\]

respectively, where

\[
\sigma_0 = \frac{3E}{2R}, \quad h(c) = \frac{1 + \lambda}{2R} c^2 \frac{3 \Delta \gamma}{2 \sigma_0}.
\]

As a result, the traction-separation relation can be expressed as

\[
\frac{p_A}{\sigma_0} = -\left( 1 - \frac{h}{h(c)} \right)^{1/2}.
\]

To investigate the dependence of the adhesive traction on the surface separation within the cohesive zone, Fig. 9 plots the variations in the traction-separation relation with different cohesive zone sizes (\( c = 1.2a \) and \( c = 3a \)). In this figure the surface separation is normalized by \( h_c \) defined in Eq. (5.4). The Lenhard–Jones and the M–D traction-separation laws are also shown for comparison. As approximations of the more realistic Lennard–Jones law, the D-H
and work of adhesion with the same based on these results, a complete transition between the JKR applied load, contact half-width and the size of cohesive zone. traction, deformation field and the equilibrium relation among Closed-form analytical solutions are obtained for the interfacial Fig. 9. Variations of the traction-separation relation with different cohesive zone maximum traction capability of different models. JKR and Hertz type solutions are given by Eqs. (5.3) and (5.10), respectively.

6. Conclusion

The present paper provides an alternative cohesive zone solution for 2-D adhesive cylindrical contact by extending the double–Hertz model of Greenwood and Johnson. This is achieved by describing the adhesive force in terms of the difference between two Hertzian pressures corresponding to different contact widths. Closed-form analytical solutions are obtained for the interfacial traction, deformation field and the equilibrium relation among applied load, contact half-width and the size of cohesive zone. Based on these results, a complete transition between the JKR and the Hertz type contact models is captured by defining a dimensionless transition parameter $\mu$, which governs the range of applicability of different models. JKR and Hertz type solutions are included as two limiting cases of the present model. An interesting finding is that unlike the 3-D case, the Hertz type solution instead of DMT type solution is recovered for the case of small $\mu$. In fact, this was also found in the M–D solution (Baney and Hui, 1997; Johnson and Greenwood, 2008), which is due to the fact that the adhesion forces scale with the characteristic contact size in quite different ways under the 2-D and 3-D cases, respectively. The present work laid a foundation for investigating other complex adhesive cylindrical contact problems involving rough surfaces, viscoelastic materials and non-homogeneous materials. Corresponding results will be reported elsewhere.

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