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On Onsager relations and linear electromagnetic materials

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Abstract

We investigated the Onsager relations in the context of electromagnetic constitutive relations of linear, homogeneous materials. We determined that application of the Onsager relations to the constitutive equations relating **P** and **M** to both **E** and **B** is in accord with Lorentz reciprocity as well as the Post constraint. Our conclusions are particularly significant for research on linear magnetoelectric materials.

Keywords: Linear materials; Macroscopic reciprocity; Magnetoelectric materials; Microscopic reversibility; Onsager relations

1. Introduction

In two seminal papers published in 1931 [1,2], with the assumption of microscopic reversibility, Onsager derived a set of reciprocity relations applicable to coupled linear phenomenons at macroscopic length scales. Fourteen years later, Casimir [3] improved the foundations of the Onsager relations. Initially considered applicable to purely instantaneous phenomenons—or, at least, when "time-lag can be neglected" [1, p. 419]—the Onsager relations widened in scope as a result of the fluctuation–dissipation theorem [4] to time-harmonic phenomenons [5]. Sections 123–125 of the famous textbook of Landau and Lifshitz on statistical physics provide a lucid introduction to the Onsager relations [6], but we also recommend a perusal of a classic monograph by de Groot [7]. A modern appraisal has been provided by Berdichevsky [8], whose paper motivated the work leading to this communication.

Our focus is the *correct* application of the Onsager relations for linear electromagnetic materials. This issue can be traced back to a 1973 paper by Rado [9]. This paper contains a major conflict between a consequence of the assumption of material response without any delay whatsoever and the Onsager relations as expounded by Callen et al. [5]. The former is definitely a noncausal assumption in electromagnetism [10,11], leading to false symmetries between the electromagnetic constitutive parameters [12]. Furthermore, Rado considered **E** and **H** as primitive fields, but **E** and **B** are taken to be the primitive fields in *modern electromagnetism* [13–15]. To the best of our knowledge, no other *original* investigation of the Onsager relations in electromagnetism exists.

Due to the currently increasing emphasis on engineered nanomaterials [16,17] and complex electromagnetic materials [18,19], it is imperative that the application of fundamental principles (such as the Onsager relations) be carefully examined with modern terminology. Accordingly, in the following sections, we first review the Onsager relations in general. Then we apply the Onsager relations to the electromagnetic constitutive relations of linear, homogeneous,

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bianisotropic materials. We show that a naïve application to constitutive equations relating **D** and **H** to both **E** and **B** yields unphysical results, but that application to constitutive equations relating **P** and **M** to both **E** and **B** is in accord with Lorentz reciprocity [20] as well as the Post constraint [21,22].

2. Onsager relations

Let us consider the linear macroscopic constitutive

$$L_m = \sum_{n=1}^{N} \Phi_{mn} F_n, \quad m \in [1, N], \tag{1}$$

where $N > 1$, L_m are the *Onsager fluxes* and F_m are the *Onsager forces*. The Onsager relations deal with the constitutive parameters Φ_{mn}.

The derivation of the Onsager relations proceeds with the postulation of N state variables a_n, $n \in [1, N]$. The state variables are divided into two groups. The first $\tilde{N} \leqslant N$ state variables are supposed to be *even* and the remaining $N - \tilde{N}$ state variables are supposed to be *odd* with respect to a reversal of velocities of the microscopic particles constituting the linear medium; in other words,

$$\overline{a_m(t)a_n(t+\tau)} = \overline{a_m(t)a_n(t-\tau)}, \quad \text{if} \begin{cases} m \in [1, \tilde{N}] \text{ and } n \in [1, \tilde{N}] \\ \text{or} \\ m \in [\tilde{N}+1, N] \text{ and } n \in [\tilde{N}+1, N] \end{cases} \tag{2}$$

and

$$\overline{a_m(t)a_n(t+\tau)} = -\overline{a_m(t)a_n(t-\tau)}, \quad \text{if} \begin{cases} m \in [1, \tilde{N}] \text{ and } n \in [\tilde{N}+1, N] \\ \text{or} \\ m \in [\tilde{N}+1, N] \text{ and } n \in [1, \tilde{N}], \end{cases} \tag{3}$$

where the overbar indicates averaging over time t [3].

In terms of the state variables, the Onsager fluxes are defined as

$$L_m = \frac{\partial}{\partial t} a_m, \quad m \in [1, N]; \tag{4}$$

the *Onsager forces* are defined as

$$F_m = -\sum_{n=1}^{N} g_{mn} a_n, \quad m \in [1, N]; \tag{5}$$

and the coefficients g_{mn} help define the deviation ΔS of the entropy from its equilibrium value as the quadratic expression [7]

$$\Delta S = -\frac{1}{2} \sum_{m=1}^{\tilde{N}} \sum_{n=1}^{\tilde{N}} g_{mn} a_m a_n - \frac{1}{2} \sum_{m=\tilde{N}+1}^{N} \sum_{n=\tilde{N}+1}^{N} g_{mn} a_m a_n. \tag{6}$$

In consequence of the microscopic reversibility indicated by (2) and (3), the constitutive parameters satisfy the Onsager relations

$$\Phi_{mn} = \Phi_{nm}, \quad \text{if} \begin{cases} m \in [1, \tilde{N}] \text{ and } n \in [1, \tilde{N}] \\ \text{or} \\ m \in [\tilde{N}+1, N] \text{ and } n \in [\tilde{N}+1, N] \end{cases} \tag{7}$$

and

$$\Phi_{mn} = -\Phi_{nm}, \quad \text{if} \begin{cases} m \in [1, \tilde{N}] \text{ and } n \in [\tilde{N}+1, N] \\ \text{or} \\ m \in [\tilde{N}+1, N] \text{ and } n \in [1, \tilde{N}]. \end{cases} \tag{8}$$

In an external magnetostatic field $\mathbf{B}_{dc}$, (7) and (8) are modified to

$$\Phi_{mn}(\mathbf{B}_{dc}) = \Phi_{nm}(-\mathbf{B}_{dc}), \quad \text{if} \begin{cases} m \in [1, \tilde{N}] \text{ and } n \in [1, \tilde{N}] \\ \text{or} \\ m \in [\tilde{N}+1, N] \text{ and } n \in [\tilde{N}+1, N] \end{cases} \tag{9}$$

and

$$\Phi_{mn}(\mathbf{B}_{dc}) = -\Phi_{nm}(-\mathbf{B}_{dc}), \quad \text{if} \begin{cases} m \in [1, \tilde{N}] \text{ and } n \in [\tilde{N}+1, N] \\ \text{or} \\ m \in [\tilde{N}+1, N] \text{ and } n \in [1, \tilde{N}], \end{cases} \tag{10}$$

respectively.

3. Application to linear electromagnetism

3.1. Constitutive equations for D and H

Let us now consider a linear, homogeneous, bianisotropic medium. Its constitutive equations can be written in a Cartesian coordinate system as

$$\left.\begin{aligned} D_j &= \textstyle\sum_{k=1}^{3} \varepsilon_{jk} \circ E_k + \xi_{jk} \circ B_k \\ H_j &= \textstyle\sum_{k=1}^{3} \zeta_{jk} \circ E_k + \nu_{jk} \circ B_k \end{aligned}\right\}, \quad j \in [1, 3]. \tag{11}$$

We have adopted here the modern view of electromagnetism, wherein **E** and **B** are the primitive fields while **D** and **H** are the induction fields [13–15]. The operation $\circ$ indicates a temporal convolution operation in the time domain, and simple multiplication in the frequency domain [23].

Now, **D** and **E** are even, but **H** and **B** are odd, with respect to time-reversal. With that in mind, we can rewrite (11) compactly as

$$Q_m = \sum_{n=1}^{N} \Lambda_{mn} \circ F_n, \quad m \in [1, N], \tag{12}$$

where $F_m = E_m$, $F_{m+3} = B_m$, $Q_m = D_m$ and $Q_{m+3} = H_m$ for $m \in [1, 3]$; furthermore, $\tilde{N} = 3$ and $N = 6$.

With the assumption of microscopic reversibility, application of the Onsager relations (9) and (10) yields the following symmetries:

$$\begin{aligned} &\Lambda_{mn}(\mathbf{B}_{dc}) = \Lambda_{nm}(-\mathbf{B}_{dc}), \quad m \in [1,3], \quad n \in [1,3], \\ &\Lambda_{mn}(\mathbf{B}_{dc}) = \Lambda_{nm}(-\mathbf{B}_{dc}), \quad m \in [4,6], \quad n \in [4,6], \\ &\Lambda_{mn}(\mathbf{B}_{dc}) = -\Lambda_{nm}(-\mathbf{B}_{dc}), \quad m \in [1,3], \quad n \in [4,6]. \end{aligned} \tag{13}$$

Eqs. (13) imply that

$$\begin{aligned} &\varepsilon_{jk}(\mathbf{B}_{dc}) = \varepsilon_{kj}(-\mathbf{B}_{dc}), \\ &v_{jk}(\mathbf{B}_{dc}) = v_{kj}(-\mathbf{B}_{dc}), \\ &\xi_{jk}(\mathbf{B}_{dc}) = -\zeta_{kj}(-\mathbf{B}_{dc}). \end{aligned} \tag{14}$$

3.2. Constitutive equations for P and M

When considering a material medium, as distinct from matter-free space (i.e. vacuum), the presence of matter is indicated by the polarization $\mathbf{P} = \mathbf{D} - \varepsilon_o \mathbf{E}$ and the magnetization $\mathbf{M} = \mu_o^{-1}\mathbf{B} - \mathbf{H}$, where ε_o and μ_o are the permittivity and the permeability of matter-free space. Linear constitutive equations for **P** and **M** can be stated as

$$\left.\begin{aligned} P_j = \textstyle\sum_{k=1}^{3} \chi_{jk}^{(1)} \circ E_k + \chi_{jk}^{(2)} \circ B_k \\ M_j = \textstyle\sum_{k=1}^{3} \chi_{jk}^{(3)} \circ E_k + \chi_{jk}^{(4)} \circ B_k, \end{aligned}\right\}, \quad j \in [1,3], \tag{15}$$

where

$$\begin{aligned} &\varepsilon_{jk} = \varepsilon_o \delta_{jk} + \chi_{jk}^{(1)}, \\ &v_{jk} = \mu_o^{-1} \delta_{jk} - \chi_{jk}^{(4)}, \\ &\xi_{jk} = \chi_{jk}^{(2)}, \\ &\zeta_{jk} = -\chi_{jk}^{(3)}, \end{aligned} \tag{16}$$

and δ_{jk} is the Kronecker delta function.

As **P** is even but **M** is odd with respect to time-reversal, we can rewrite (15) as

$$R_m = \sum_{n=1}^{N} \Psi_{mn} \circ F_n, \quad m \in [1, N], \tag{17}$$

where $R_m = P_m$ and $R_{m+3} = M_m$ for $m \in [1,\ 3]$. As the microscopic processes underlying the constitutive parameters in (17) are reversible, Ψ_{mn} must satisfy (9) and (10); thus,

$$\begin{aligned} &\Psi_{mn}(\mathbf{B}_{dc}) = \Psi_{nm}(-\mathbf{B}_{dc}), \quad m \in [1,3], \quad n \in [1,3], \\ &\Psi_{mn}(\mathbf{B}_{dc}) = \Psi_{nm}(-\mathbf{B}_{dc}), \quad m \in [4,6], \quad n \in [4,6], \\ &\Psi_{mn}(\mathbf{B}_{dc}) = -\Psi_{nm}(-\mathbf{B}_{dc}), \quad m \in [1,3], \quad n \in [4,6], \end{aligned} \tag{18}$$

whence the symmetries

$$\begin{aligned} &\chi_{jk}^{(1)}(\mathbf{B}_{dc}) = \chi_{kj}^{(1)}(-\mathbf{B}_{dc}), \\ &\chi_{jk}^{(4)}(\mathbf{B}_{dc}) = \chi_{kj}^{(4)}(-\mathbf{B}_{dc}), \\ &\chi_{jk}^{(2)}(\mathbf{B}_{dc}) = -\chi_{kj}^{(3)}(-\mathbf{B}_{dc}), \end{aligned} \tag{19}$$

are predicted by the Onsager relations as the macroscopic consequences of microscopic reversibility.

3.3. The conflict

Eqs. (19) imply that

$$\begin{aligned} &\varepsilon_{jk}(\mathbf{B}_{dc}) = \varepsilon_{kj}(-\mathbf{B}_{dc}), \\ &v_{jk}(\mathbf{B}_{dc}) = v_{kj}(-\mathbf{B}_{dc}), \\ &\xi_{jk}(\mathbf{B}_{dc}) = \zeta_{kj}(-\mathbf{B}_{dc}), \end{aligned} \tag{20}$$

by virtue of (16).

But $(20)_3$ disagrees completely with $(14)_3$. Let us reiterate that both (14) and (20) come about from the application of the Onsager relations, contingent upon the assumption of microscopic reversibility. Yet, at most, only one of the two must be correct.

3.4. Resolution of the conflict

Onsager's own papers help resolve the conflict. His papers were concerned with motion of microscopic particles, and he considered his work to hold true for heat conduction, gaseous diffusion and related transport problems. The Onsager forces must be causative agents, while the Onsager fluxes must be directly concerned with particulate motion. This understanding is reinforced by subsequent commentaries [6,7].

Therefore, in order to *correctly* exploit the Onsager relations in electromagnetics, we must isolate those parts of **D** and **H** which indicate the presence of a *material*, because microscopic processes cannot occur in matter-free space (i.e. vacuum). The matter-indicating parts of **D** and **H** are **P** and **M**. *Hence*, (20) *must be accepted and* (14) *must be discarded*.

With $\mathbf{B}_{dc} = \mathbf{0}$, the symmetries (20) coincide—unlike (14)—with those mandated by Lorentz reciprocity [20, Eqs. 23]. Also unlike (14), the symmetries (20) are compatible with the Post constraint [21,22]

$$\sum_{j=1}^{3} \xi_{jj} = \sum_{j=1}^{3} \zeta_{jj} \tag{21}$$

which must be satisfied by all (i.e. Lorentz-reciprocal as well as Lorentz-nonreciprocal) linear materials. These two well-known facts also support our decision to discard (14) in favor of (20).

The literature on linear magnetoelectric materials is replete with the use of (14), derived most prominently by O'Dell [24, Eq. 2.64]; and Rado [9] appears to have distorted his initial results in order to confirm to that derivation. Thus, the impact of the correct application of the Onsager relations should be felt mostly in research on magnetoelectric materials [12]. A secondary impact shall be on the inadequately measured properties of the so-called Tellegen medium and Tellegen particles, a review of which is available elsewhere [22, Section 5].

4. Concluding remarks

In this communication, we first reviewed the Onsager relations which delineate the macroscopic consequences of microscopic reversibility in linear materials. Then we applied the relations to the electromagnetic constitutive relations of homogeneous bianisotropic materials. We determined that a naïve application to constitutive equations relating **D** and **H** to both **E** and **B** yields unphysical results, but that application to constitutive equations relating **P** and **M** to both **E** and **B** is in accord with Lorentz reciprocity as well as the Post constraint.

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On the genesis of Post constraint in modern electromagnetism

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Abstract: The genesis of the Post constraint is premised on two attributes of modern electromagnetism: (i) its microscopic nature, and (ii) the status of $\tilde{\boldsymbol{e}}(\boldsymbol{x}, t)$ and $\tilde{\boldsymbol{b}}(\boldsymbol{x}, t)$ as the primitive electromagnetic fields. This constraint can therefore not arise in EH-electromagnetism, wherein the primitive electromagnetic fields are the macroscopic fields $\tilde{\boldsymbol{E}}(\boldsymbol{x}, t)$ and $\tilde{\boldsymbol{H}}(\boldsymbol{x}, t)$. Available experimental evidence against the Post constraint is incomplete and inconclusive.

Key words: Electromagnetic theories – free space – macroscopic physics – magnetoelectric materials – microphysics – Post constraint – tellegen medium

1. Introduction

Ever since its enunciation in 1962 [1], the Post constraint has been an enigma. It was ignored for over three decades by the electromagnetics community for reasons that will probably be extracted only by future historians of science. It arose from obscurity like a phoenix in 1994 in the context of linear, nonreciprocal, biisotropic mediums [2], and since then has been the subject of discussion in the complex-mediums electromagnetics research community.

A remarkable feature of the Post constraint is that it permits a sharp distinction between two widely prevalent conceptions of electromagnetic phenomenons. The genesis of the Post constraint lies in the microphysical basis of *modern electromagnetism*, whereby the (necessarily macroscopic) constitutive functions must be conceived as piecewise homogeneous entities and can therefore not vary continuously in spacetime. In contrast, *EH-electromagnetism* is essentially macroscopic, and its principles seem to be inimical to the validity of the Post constraint. Available experimental evidence does not negate the Post constraint, but cannot be held to be conclusive either.

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These issues are discussed in this essay. Section 2 is an exposition of modern electromagnetism encompassing both the microscopic and the macroscopic levels. Section 3 presents the rationale for and the genesis of the Post constraint. The characteristics of EH-electromagnetism relevant to the Post constraint are given in Section 4, while experimental evidence is reviewed in Section 5. Finally, in Section 6 the constitutive equations of free space are deduced in relation to the Post constraint.

2. Modern electromagnetism

Electromagnetism today is a microscopic science, even though it is mostly used in its macroscopic form. It was certainly a macroscopic science when Maxwell unified the equations of Coulomb, Gauss, Faraday, and Ampère, added a displacement current to Ampère's equation, and produced the four equations to which his name is attached. Although Maxwell had abandoned a mechanical basis for electromagnetism during the early 1860s, and even used terms like *molecular vortices*, a close reading [3] of his papers will convince the reader that Maxwell's conception of electromagnetism – like that of most of his contemporaries – was macroscopic.

By the end of the 19th century, that conception had been drastically altered [4]. Hall's successful explanation of the eponymous effect, the postulation of the electron by Stoney and its subsequent discovery by Thomson, and Larmor's theory of the electron precipitated that alteration. It was soon codified by Lorentz and Heaviside, so that the 20th century dawned with the acquisition of a microphysical basis by electromagnetism. Maxwell's equations remained unaltered in form at macroscopic length scales, but their roots now lie in the fields engendered by microscopic charge quantums. The subsequent emergence of *quantum mechanics* did not change the form of the macroscopic equations either, although the notion of a field lost its determinism and an inherent uncertainty was recognized in the measurements of key variables [5].

2.1. Microscopic Maxwell postulates

The microscopic fields are just two: the electric field $\tilde{\boldsymbol{e}}(\boldsymbol{x},t)$ and the magnetic field $\tilde{\boldsymbol{b}}(\boldsymbol{x},t)$.[1] These two are accorded the status of primitive fields in modern electromagnetism. Both fields vary extremely rapidly as functions of position $\boldsymbol{x}$ and time t. Their sources are the microscopic charge density $\tilde{c}(\boldsymbol{x},t)$ and the microscopic current density $\tilde{\boldsymbol{j}}(\boldsymbol{x},t)$, where

$$\tilde{c}(\boldsymbol{x},t) = \sum_\ell q_\ell\, \delta[\boldsymbol{x} - \boldsymbol{x}_\ell(t)]\ , \tag{1}$$

$$\tilde{\boldsymbol{j}}(\boldsymbol{x},t) = \sum_\ell q_\ell \boldsymbol{v}_\ell\, \delta[\boldsymbol{x} - \boldsymbol{x}_\ell(t)]\,; \tag{2}$$

$\delta(\cdot)$ is the Dirac delta function; while $\boldsymbol{x}_\ell(t)$ and $\boldsymbol{v}_\ell(t)$ are the position and the velocity of the point charge q_ℓ. Uncertainties in the measurements of the positions and the velocities of the discrete point charges open the door to quantum mechanics, but we need not traverse that path here.

All of the foregoing fields and sources appear in the *microscopic* Maxwell postulates:

$$\nabla\cdot\tilde{\boldsymbol{e}}(\boldsymbol{x},t) = \varepsilon_0^{-1}\tilde{c}(\boldsymbol{x},t)\ , \tag{3}$$

$$\nabla\times\tilde{\boldsymbol{b}}(\boldsymbol{x},t) - \varepsilon_0\mu_0\,\frac{\partial}{\partial t}\,\tilde{\boldsymbol{e}}(\boldsymbol{x},t) = \mu_0\tilde{\boldsymbol{j}}(\boldsymbol{x},t)\,, \tag{4}$$

$$\nabla\cdot\tilde{\boldsymbol{b}}(\boldsymbol{x},t) = 0\,, \tag{5}$$

$$\nabla\times\tilde{\boldsymbol{e}}(\boldsymbol{x},t) + \frac{\partial}{\partial t}\,\tilde{\boldsymbol{b}}(\boldsymbol{x},t) = \boldsymbol{0}\,. \tag{6}$$

In these equations and hereafter, $\varepsilon_0 = 8.854\times10^{-12}$ F/m and $\mu_0 = 4\pi\times10^{-7}$ H/m are the permittivity and the permeability of free space (i.e., vacuum), respectively. The first two postulates are inhomogeneous differential equations as they contain source terms on their right sides, while the last two are homogeneous differential equations.

2.2. Macroscopic Maxwell postulates

Macroscopic measuring devices average over (relatively) large spatial and temporal intervals. Therefore, spatiotemporal averaging of the microscopic quantities appears necessary in order to deduce the *macroscopic* Maxwell postulates from (3)–(6). Actually, only spatial averaging is necessary [6], because it implies temporal averaging due to the finite magnitude of the universal maximum speed $(\varepsilon_0\mu_0)^{-1/2}$. Denoting the macroscopic charge and current densities, respectively, by $\tilde{\varrho}(\boldsymbol{x},t)$ and $\tilde{\boldsymbol{J}}(\boldsymbol{x},t)$, we obtain the *macroscopic* Maxwell postulates

$$\nabla\cdot\tilde{\boldsymbol{E}}(\boldsymbol{x},t) = \varepsilon_0^{-1}\,\tilde{\varrho}(\boldsymbol{x},t)\,, \tag{7}$$

$$\nabla\times\tilde{\boldsymbol{B}}(\boldsymbol{x},t) - \varepsilon_0\mu_0\frac{\partial}{\partial t}\,\tilde{\boldsymbol{E}}(\boldsymbol{x},t) = \mu_0\,\tilde{\boldsymbol{J}}(\boldsymbol{x},t)\,, \tag{8}$$

$$\nabla\cdot\tilde{\boldsymbol{B}}(\boldsymbol{x},t) = 0\,, \tag{9}$$

$$\nabla\times\tilde{\boldsymbol{E}}(\boldsymbol{x},t) + \frac{\partial}{\partial t}\,\tilde{\boldsymbol{B}}(\boldsymbol{x},t) = \boldsymbol{0}\,, \tag{10}$$

which involve the macroscopic primitive fields $\tilde{\boldsymbol{E}}(\boldsymbol{x},t)$ and $\tilde{\boldsymbol{B}}(\boldsymbol{x},t)$ as the spatial averages of $\tilde{\boldsymbol{e}}(\boldsymbol{x},t)$ and $\tilde{\boldsymbol{b}}(\boldsymbol{x},t)$, respectively. From (7) and (8), a macroscopic continuity equation for the source densities can be derived as

$$\nabla\cdot\tilde{\boldsymbol{J}}(\boldsymbol{x},t) + \frac{\partial}{\partial t}\,\tilde{\varrho}(\boldsymbol{x},t) = 0\,. \tag{11}$$

2.3. Familiar form of macroscopic Maxwell postulates

Equations (7)–(10) are not the familiar form of the macroscopic Maxwell postulates, even though they hold in free space as well as in matter. The familiar form emerges after the recognition that matter contains, in general, both free charges and bound charges. Free and bound source densities can be separated as

$$\tilde{\varrho}(\boldsymbol{x},t) = \tilde{\varrho}_{\rm so}(\boldsymbol{x},t) - \nabla\cdot\tilde{\boldsymbol{P}}(\boldsymbol{x},t) \tag{12}$$

and

$$\tilde{\boldsymbol{J}}(\boldsymbol{x},t) = \tilde{\boldsymbol{J}}_{\rm so}(\boldsymbol{x},t) + \frac{\partial}{\partial t}\,\tilde{\boldsymbol{P}}(\boldsymbol{x},t) + \nabla\times\tilde{\boldsymbol{M}}(\boldsymbol{x},t)\,. \tag{13}$$

This decomposition is consistent with (11), provided the free source densities obey the reduced continuity equation

$$\nabla\cdot\tilde{\boldsymbol{J}}_{\rm so}(\boldsymbol{x},t) + \frac{\partial}{\partial t}\,\tilde{\varrho}_{so}(\boldsymbol{x},t) = 0\,. \tag{14}$$

The free source densities represent "true" sources which can be externally impressed. Whereas $\tilde{\boldsymbol{J}}_{\rm so}(\boldsymbol{x},t)$ is the source current density, $\tilde{\varrho}_{\rm so}(\boldsymbol{x},t)$ is the source charge density.

Bound source densities represent matter in its macroscopic form and are, in turn, quantified by the polarization $\tilde{\boldsymbol{P}}(\boldsymbol{x},t)$ and the magnetization $\tilde{\boldsymbol{M}}(\boldsymbol{x},t)$. Both $\tilde{\boldsymbol{P}}(\boldsymbol{x},t)$ and $\tilde{\boldsymbol{M}}(\boldsymbol{x},t)$ are nonunique to the extent that they can be replaced by $\tilde{\boldsymbol{P}}(\boldsymbol{x},t) - \nabla\times\tilde{\boldsymbol{A}}(\boldsymbol{x},t)$ and $\tilde{\boldsymbol{M}}(\boldsymbol{x},t) + (\partial/\partial t)\,\tilde{\boldsymbol{A}}(\boldsymbol{x},t)$, respectively, in (12) and (13) without affecting the left sides of either equation.

Polarization and magnetization are subsumed in the definitions of the electric induction $\tilde{\boldsymbol{D}}(\boldsymbol{x},t)$ and the magnetic induction $\tilde{\boldsymbol{H}}(\boldsymbol{x},t)$ as follows:

$$\tilde{\boldsymbol{D}}(\boldsymbol{x},t) = \varepsilon_0\,\tilde{\boldsymbol{E}}(\boldsymbol{x},t) + \tilde{\boldsymbol{P}}(\boldsymbol{x},t)\,, \tag{15}$$

$$\tilde{\boldsymbol{H}}(\boldsymbol{x},t) = \mu_0^{-1}\,\tilde{\boldsymbol{B}}(\boldsymbol{x},t) - \tilde{\boldsymbol{M}}(\boldsymbol{x},t)\,. \tag{16}$$

Then, (7)–(10) metamorphose into the familiar form of the macroscopic Maxwell postulates:

$$\nabla\cdot\tilde{\boldsymbol{D}}(\boldsymbol{x},t) = \tilde{\varrho}_{\rm so}(\boldsymbol{x},t)\,, \tag{17}$$

$$\nabla\times\tilde{\boldsymbol{H}}(\boldsymbol{x},t) - \frac{\partial}{\partial t}\,\tilde{\boldsymbol{D}}(\boldsymbol{x},t) = \tilde{\boldsymbol{J}}_{\rm so}(\boldsymbol{x},t)\,, \tag{18}$$

$$\nabla\cdot\tilde{\boldsymbol{B}}(\boldsymbol{x},t) = 0\,, \tag{19}$$

$$\nabla\times\tilde{\boldsymbol{E}}(\boldsymbol{x},t) + \frac{\partial}{\partial t}\,\tilde{\boldsymbol{B}}(\boldsymbol{x},t) = \boldsymbol{0}\,. \tag{20}$$

Let us note, in passing, that the fields $\tilde{\boldsymbol{d}}(\boldsymbol{x},t)$ and $\tilde{\boldsymbol{h}}(\boldsymbol{x},t)$ do not exist in microphysics, matter being an ensemble of point charges in free space.

[1]) The lower-case letter signifies that the quantity is microscopic, while the tilde ˜ indicates dependence on time.

2.4. Linear constitutive relations

The induction fields at some point in spacetime $(\boldsymbol{x}, t)$ can depend *locally* on the primitive fields at the same $(\boldsymbol{x}, t)$. This dependence can be spatially nonhomogeneous (i.e., dependent on space $\boldsymbol{x}$) and/or can vary with time t (i.e., age). In addition, the induction fields at $(\boldsymbol{x}, t)$ can depend nonlocally on the primitive fields at some $(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)$, where the spacetime interval $(\boldsymbol{x}_h, t_h)$, $t_h \geq 0$, must be timelike in order to be causally influential [7, pp. 85–89]. Thus, the most general linear constitutive relations [8]

$$\tilde{\boldsymbol{D}}(\boldsymbol{x},t) = \int\int \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h + \int\int \tilde{\boldsymbol{\xi}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h \tag{21}$$

and

$$\tilde{\boldsymbol{H}}(\boldsymbol{x},t) = \int\int \tilde{\boldsymbol{\zeta}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h + \int\int \tilde{\boldsymbol{\nu}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h \tag{22}$$

can describe any linear medium – indeed, the entire universe after linearization, to which are the developments in Section 3 applied. The integrals extend only over the causal values of $(\boldsymbol{x}_h, t_h)$, but that does not restrict the analysis presented here. We also assume an inertial frame of reference hereafter.

3. The Post constraint

Four second-rank tensors appear in the foregoing constitutive relations: $\tilde{\boldsymbol{\varepsilon}}$ is the permittivity tensor, $\tilde{\boldsymbol{\nu}}$ is the impermeability tensor, while $\tilde{\boldsymbol{\xi}}$ and $\tilde{\boldsymbol{\zeta}}$ are the magnetoelectric tensors. Together, these four tensors contain 36 scalar functions; but the Post constraint indicates that only 35, at most, are independent. This was clarified elsewhere [9] using 4-tensor notation, but we revisit the issue here for completeness. Let us therefore express the magnetoelectric tensors as

$$\tilde{\boldsymbol{\xi}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) = \tilde{\boldsymbol{\alpha}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) + \tfrac{1}{6}\,\mathbf{I}\tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \tag{23}$$

and

$$\tilde{\boldsymbol{\zeta}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) = \tilde{\boldsymbol{\beta}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) - \tfrac{1}{6}\,\mathbf{I}\,\tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\,, \tag{24}$$

where $\mathbf{I}$ is the identity tensor and the scalar function

$$\tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) = \mathrm{Trace}\,\left(\tilde{\boldsymbol{\xi}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) - \tilde{\boldsymbol{\zeta}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\right). \tag{25}$$

Therefore,

$$\mathrm{Trace}\,\left(\tilde{\boldsymbol{\alpha}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) - \tilde{\boldsymbol{\beta}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\right) \equiv 0\,. \tag{26}$$

3.1. Rationale for the Post constraint

Let us recall that (19) and (20) do not contain the induction fields $\tilde{\boldsymbol{D}}(\boldsymbol{x},t)$ and $\tilde{\boldsymbol{H}}(\boldsymbol{x},t)$. Hence, (21) and (22) must be substituted only in (17) and (18); thus,

$$\begin{aligned}
&\int\int \nabla \cdot \left(\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right.\\
&\quad \left.+ \tilde{\boldsymbol{\alpha}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right)\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&\quad + \tfrac{1}{6}\int\int \tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\,\left(\nabla \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right)\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&\quad + \tfrac{1}{6}\int\int \left(\nabla\tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\right) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,t_h\\
&= \tilde{\varrho}_{\mathrm{so}}(\boldsymbol{x},t)
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\int\int \nabla \times \;\left(\tilde{\boldsymbol{\beta}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right.\\
&\quad \left.+ \tilde{\boldsymbol{\nu}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right)\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&\quad - \int\int \frac{\partial}{\partial t}\,\left(\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right.\\
&\quad \left.+ \tilde{\boldsymbol{\alpha}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right)\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&\quad - \frac{1}{6}\int\int \tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\,\Big(\nabla \times \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\\
&\quad + \frac{\partial}{\partial t}\,\tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\Big)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&\quad - \frac{1}{6}\int\int \left(\nabla\tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\right) \times \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&\quad - \frac{1}{6}\int\int \left(\frac{\partial}{\partial t}\,\tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\right) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&= \tilde{\boldsymbol{J}}_{\mathrm{so}}(\boldsymbol{x},t)\,.
\end{aligned} \tag{28}$$

The second integral on the left side of (27) is null-valued by virtue of (19); likewise, the third integral on the left side of (28) is null-valued by virtue of (20). Therefore, the four macroscopic Maxwell postulates now read as follows:

$$\begin{aligned}
&\int\int \nabla \cdot \left(\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right.\\
&\quad \left.+ \tilde{\boldsymbol{\alpha}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right)\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&\quad + \tfrac{1}{6}\,\int\int \left(\nabla\tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\right) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&= \tilde{\varrho}_{so}(\boldsymbol{x},t)\,,
\end{aligned} \tag{29}$$

$$\begin{aligned}
&\int\int \nabla \times \left(\tilde{\boldsymbol{\beta}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right.\\
&\quad \left.+ \tilde{\boldsymbol{\nu}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right)\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&\quad - \int\int \frac{\partial}{\partial t}\,\left(\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right.\\
&\quad \left.+ \tilde{\boldsymbol{\alpha}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\right)\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&\quad - \frac{1}{6}\int\int \left(\nabla\tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\right) \times \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&\quad - \frac{1}{6}\int\int \left(\frac{\partial}{\partial t}\,\tilde{\Psi}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h)\right) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}-\boldsymbol{x}_h, t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\\
&= \tilde{\boldsymbol{J}}_{\mathrm{so}}(\boldsymbol{x},t)\,,
\end{aligned} \tag{30}$$

$$\nabla \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}, t) = 0\,, \tag{31}$$

$$\nabla \times \tilde{\boldsymbol{E}}(\boldsymbol{x}, t) + \frac{\partial}{\partial t}\tilde{\boldsymbol{B}}(\boldsymbol{x}, t) = \boldsymbol{0}\,. \tag{32}$$

Differentiation of the product of two functions is distributive. Hence, the thirty-five independent constitutive scalars in $\tilde{\boldsymbol{\varepsilon}}$, $\tilde{\boldsymbol{\alpha}}$, $\tilde{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\nu}}$ occur in (29)–(32) in two ways: (i) by themselves, and (ii) through their space- and time-derivatives. In contrast, the thirty-sixth constitutive scalar $\tilde{\Psi}$ does not occur in (29)–(32) by itself. Thus, $\tilde{\Psi}$ vanished from the macroscopic Maxwell postulates like the Cheshire cat, but left behind its derivatives like the cat's grin.

This is an anomalous situation, and its elimination leads to the Post constraint.

3.2. Post's conclusions

In a seminal contribution on the covariant structure of modern electromagnetism [1], Post made a distinction between functional and structural fields. Functional fields specify the state of a medium, and are exemplified by $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$. Structural fields, exemplified by the constitutive tensors, specify the properties of the medium. Formulating the Lagrangian and examining its Eulerian derivative [1, eq. 5.31], Post arrived at the conclusion that

$$\tilde{\Psi}(\boldsymbol{x}, t; \boldsymbol{x}_h, t_h) \equiv 0 \tag{33}$$

even for nonhomogeneous mediums [1, p. 130]. Furthermore, he held that the space- and time-derivatives of $\tilde{\Psi}(\boldsymbol{x}, t; \boldsymbol{x}_h, t_h)$ are also identically zero, so that [1, p. 129]

$$\left.\begin{array}{l} \nabla\tilde{\Psi}(\boldsymbol{x}, t; \boldsymbol{x}_h, t_h) \equiv 0 \\ \dfrac{\partial}{\partial t}\,\tilde{\Psi}(\boldsymbol{x}, t; \boldsymbol{x}_h, t_h) \equiv 0 \end{array}\right\}. \tag{34}$$

Eqs. (33) and (34) may appear to be independent but are not, because the derivatives of a constant function are zero. Eq. (33) alone is called the *Post constraint*.

3.3. Recognizable existence of $\tilde{\Psi}$

Whether $\tilde{\Psi}$ is identically null-valued or not is a moot point. The real issue is whether it has a recognizable existence or not. This stance was adopted by Lakhtakia and Weiglhofer [10].

Let us recall that all matter is microscopic. Despite the convenience proffered by continuum theories, those theories are merely approximations. Constitutive functions are macroscopic entities arising from the homogenization of assemblies of microscopic charge carriers, with free space serving as the reference medium [11]. In any small enough portion of spacetime that is homogenizable, the constitutive functions are uniform. When such a portion will be interrogated for characterization, it will have to be embedded in free space. Accordingly, the second integral on the left side of (29) as well as the third as well as the fourth integrals on the left side of (30) would vanish during the interrogation for fields inside and outside that piece. Therefore, the principle of parsimony (attributed to a 14th century monk [12]) enjoins the acceptance of (33).

3.4. Nature of the Post constraint

When linear mediums of increasing complexity are investigated, the nature of thePost constraint can appear to vary. For instance, were investigation confined to isotropic mediums [13], the condition $\tilde{\Psi} \equiv 0$ can resemble a *reciprocity constraint*. But it is not, because it does not impose any transpose-symmetry requirements on $\tilde{\boldsymbol{\varepsilon}}$, $\tilde{\boldsymbol{\alpha}}$, $\tilde{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\nu}}$ [14, eq. 23].

Another possibility is to think that the Post constraint negates the generalized duality transformation [15], but actually it does not when it is globally applied at the microscopic level [16, pp. 203–204]. Finally, the Post constraint is not a gauge transformation – i.e., a $\tilde{\Psi}$-independent field $\tilde{\boldsymbol{A}}$ cannot be found to replace $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{M}}$ by $\tilde{\boldsymbol{P}} - \nabla \times \tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{M}} + (\partial/\partial t)\,\tilde{\boldsymbol{A}}$, respectively, in order to eliminate $\tilde{\Psi}$.

The Post constraint is actually a *structural constraint*. Post may have been inspired towards it in order to eliminate a pathological constitutive relation [1, eq. 3.20], [17], and then established a covariance argument for it. Physically, this constraint arises from the following two considerations:

- The Ampère-Maxwell equation (containing the induction fields) should be independent of the Faraday equation (containing the primitive fields) at the macroscopic level, just as the two equations are mutually independent at the microscopic level.
- The constitutive functions must be characterized as piecewise uniform, being born of the spatial homogenization of microscopic entities. Therefore, if a homogeneous piece of a medium with a certain set of electromagnetic response properties cannot be recognized, the assumption of continuously nonhomogeneous analogs of that set is untenable.

4. EH-electromagnetism

Time-domain electromagnetic research is a distant second to frequency-domain electromagnetic research, as measured by the numbers of publications as well as the numbers of researchers. Much of frequency-domain research at the macroscopic level also commences with the familar form (17)–(20) of the Maxwell postulates, but the roles of $\tilde{\boldsymbol{H}}$ and $\tilde{\boldsymbol{B}}$ are interchanged [11].

Thus, constitutive relations are written to express $\tilde{\boldsymbol{D}}$ and $\tilde{\boldsymbol{B}}$ in terms of $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{H}}$. Specifically, the linear constitutive relations (21) and (22) are replaced by

$$\begin{aligned}
&\tilde{\boldsymbol{D}}(\boldsymbol{x}, t) \\
&= \int\int \tilde{\boldsymbol{\mathcal{A}}}(\boldsymbol{x}, t; \boldsymbol{x}_h, t_h) \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x} - \boldsymbol{x}_h, t - t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h \\
&\quad + \int\int \tilde{\boldsymbol{\mathcal{B}}}(\boldsymbol{x}, t; \boldsymbol{x}_h, t_h) \cdot \tilde{\boldsymbol{H}}(\boldsymbol{x} - \boldsymbol{x}_h, t - t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h
\end{aligned} \tag{35}$$

and

$$\tilde{\boldsymbol{B}}(\boldsymbol{x},t) = \int\int \tilde{\boldsymbol{\mathcal{C}}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}-\boldsymbol{x}_h,t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h + \int\int \tilde{\boldsymbol{\mathcal{D}}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \cdot \tilde{\boldsymbol{H}}(\boldsymbol{x}-\boldsymbol{x}_h,t-t_h)\,\mathrm{d}\boldsymbol{x}_h\,\mathrm{d}t_h\,, \quad (36)$$

with $\tilde{\boldsymbol{\mathcal{A}}}$, $\tilde{\boldsymbol{\mathcal{B}}}$, $\tilde{\boldsymbol{\mathcal{C}}}$ and $\tilde{\boldsymbol{\mathcal{D}}}$ as the constitutive tensors. This version of electromagnetism is called the EH-electromagnetism in this essay.

At first glance, the difference between the modern and the EH versions may not appear to be significant, particularly for linear mediums at the macroscopic level. The frequency-domain versions of the constitutive tensors $\tilde{\boldsymbol{\mathcal{A}}}$, etc., can also be microscopically motivated in much the same way as the frequency-domain versions of $\tilde{\boldsymbol{\varepsilon}}$, etc., are. Yet, there is a huge difference: The Faraday equation contains only the primitive fields while the Ampère-Maxwell equation contains only the induction fields, in modern electromagnetism, and can therefore be independent of each other just as at the microscopic level. But each of the two equations contains a primitive field and an induction field in EH-electromagnetism – hence, it is impossible for the two equations to be independent of each other at the macroscopic level. This central difference between the two versions of electromagnetism is often a source of great confusion.

4.1. Post constraint

As both the Faraday and the Ampère-Maxwell equations (at the macroscopic level) contain a primitive field and an induction field in EH-electromagnetism, it appears impossible to derive the Post constraint in the EH version. Not surprisingly, current opposition to the validity of the Post constraint invariably employs the EH version [15, 18], and older constructs that presumably invalidate the Post constraint are also based on EH-electromagnetism [19–21]. The major exception to the previous statement is the work of O'Dell [22, pp. 38–44], but it is fatally marred by the assumption of purely instantaneous – and, therefore, noncausal – constitutive relations. Simply put, *the Post constraint is valid in modern electromagnetism but probably invalid in EH-electromagnetism*.

But we hold modern electromagnetism to be *truer* than its EH counterpart [6, 23–25]. Accordingly, the Post constraint can *translated* from the former to the latter, in certain circumstances. For example, let us consider a spatially homogeneous, temporally invariant and spatially local medium: $\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) \equiv \tilde{\boldsymbol{\varepsilon}}(t_h)\,\delta(\boldsymbol{x}_h)$, etc. Employing the temporal Fourier transform[2]

$$\tilde{Z}(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\boldsymbol{x},\omega)\exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega\,, \quad (37)$$

where ω is the angular frequency and $\mathrm{i}=\sqrt{-1}$, we see that (21) and (22) transform to

$$\left.\begin{aligned} \boldsymbol{D}(\boldsymbol{x},\omega) &= \boldsymbol{\varepsilon}(\omega)\cdot\boldsymbol{E}(\boldsymbol{x},\omega) + \boldsymbol{\xi}(\omega)\cdot\boldsymbol{B}(\boldsymbol{x},\omega) \\ \boldsymbol{H}(\boldsymbol{x},\omega) &= \boldsymbol{\zeta}(\omega)\cdot\boldsymbol{E}(\boldsymbol{x},\omega) + \boldsymbol{\nu}(\omega)\cdot\boldsymbol{B}(\boldsymbol{x},\omega) \end{aligned}\right\}, \quad (38)$$

while (35) and (36) yield

$$\left.\begin{aligned} \boldsymbol{D}(\boldsymbol{x},\omega) &= \boldsymbol{\mathcal{A}}(\omega)\cdot\boldsymbol{E}(\boldsymbol{x},\omega) + \boldsymbol{\mathcal{B}}(\omega)\cdot\boldsymbol{H}(\boldsymbol{x},\omega) \\ \boldsymbol{B}(\boldsymbol{x},\omega) &= \boldsymbol{\mathcal{C}}(\omega)\cdot\boldsymbol{E}(\boldsymbol{x},\omega) + \boldsymbol{\mathcal{D}}(\omega)\cdot\boldsymbol{H}(\boldsymbol{x},\omega) \end{aligned}\right\}. \quad (39)$$

With the assumption that $\boldsymbol{\mathcal{D}}(\omega)$ is invertible, the Post constraint

$$\Psi(\omega) \equiv 0 \quad (40)$$

translates into the condition [26]

$$\mathrm{Trace}\,\left(\boldsymbol{\mathcal{B}}(\omega)\cdot\boldsymbol{\mathcal{D}}^{-1}(\omega) + \boldsymbol{\mathcal{D}}^{-1}(\omega)\cdot\boldsymbol{\mathcal{C}}(\omega)\right) \equiv 0 \quad (41)$$

for EH-electromagnetism; equivalently,

$$\mathrm{Trace}\,\left[\boldsymbol{\mathcal{D}}^{-1}(\omega)\cdot\left(\boldsymbol{\mathcal{B}}(\omega)+\boldsymbol{\mathcal{C}}(\omega)\right)\right] \equiv 0\,. \quad (42)$$

We must remember, however, that (42) is probably underivable within the framework of EH-electromagnetism, but is simply a translation of (40).

5. Experimental evidence

Fundamental questions are answered by a convergence of theoretical constructs and diverse experimental evidence. On this basis, modern electromagnetism is well-established, which provides confidence in the validity of the Post constraint. Furthermore, incontrovertible experimental results against the Post constraint are unknown. Nevertheless, the constraint is experimentally falsifiable, and available experimental evidence presented against it must not be dismissed lightly. Let us examine that evidence now.

5.1. Magnetoelectric materials

Anisotropic materials with magnetoelectric tensors are commonplace. Typically, such properties are exhibited at low frequencies and low temperatures. Although their emergence in research literature can be traced back to Pierre Curie [27], a paper published originally in 1959 [20] focused attention on them. O'Dell wrote a famous book on these materials [22] in 1970.

A significant result of O'Dell [22, eq. 2.64], although derived for spatiotemporally uniform and spatiotemporally local mediums (i.e., $\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x},t;\boldsymbol{x}_h,t_h) = \tilde{\boldsymbol{\varepsilon}}\delta(\boldsymbol{x}_h)\,\delta(t_h)$, etc.), is often used in frequency-domain literature on spatiotemporally uniform and spatially local mediums as follows:

$$\mathrm{Transpose}\,\left(\boldsymbol{\xi}(\omega)\right) = -\boldsymbol{\zeta}(\omega)\,. \quad (43)$$

---
[2]) Whereas all quantities decorated with a tilde ˜ are real-valued, their undecorated counterparts are complex-valued in general.

This equation is often held to allow materials for which $\Psi(\omega) \neq 0$. More importantly, this equation is widely used in the magnetoelectric research community to reduce experimental tedium in characterizing magnetoelectric materials. Yet this equation is based on a false premise: that materials (as distinct from free space) respond purely instantaneously [22, p. 43]. Hence, experimental data obtained after exploiting (43) cannot be trusted [28].

The false premise can be traced back to Dzyaloshinskiĭ's 1959 paper [20], wherein EH-electromagnetism was used. Astrov [29] examined the variation of $\mathcal{C}(\omega)$ of Cr_2O_3 with temperature at 10 kHz frequency. Folen et al. [30] measured $\mathcal{C}(\omega)$ of Cr_2O_3 at 1 kHz frequency and presumably equated it to $\mathcal{B}(\omega)$ by virtue of the 1959 antecedent [31], but did not actually measure $\mathcal{B}(\omega)$.[3] Rado and Folen [32, 33] verified the existences of both $\mathcal{B}(\omega)$ and $\mathcal{C}(\omega)$ for the same substance, and they also established that both quantities are temperature-dependent, but they too did not measure $\mathcal{B}(\omega)$. Similar deficiencies in other published reports have been detailed elsewhere [28]. Recently, Raab [34] has rightly called for comprehensive and complete characterization of magnetoelectric materials, with (43) not assumed in advance but actually subjected to a test.

5.2. Tellegen medium

Take a fluid medium in which permanent, orientable, electric dipoles exist in abundance. Stir in small ferromagnetic particles with permanent magnetic dipole moments, ensuring that each electric dipole moment cleaves together with a parallel magnetic dipole moment, to form a Tellegen particle [18]. Shake well for a homogeneous, isotropic suspension of Tellegen particles. This is the recipe that Tellegen [19] gave for the so-called Tellegen medium, after he had conceptualized the gyrator.

The frequency-domain constitutive relations of this medium may be set down as

$$\left.\begin{aligned} \boldsymbol{D}(\boldsymbol{x},\omega) &= \mathcal{A}(\omega)\,\boldsymbol{E}(\boldsymbol{x},\omega) + \mathcal{B}(\omega)\,\boldsymbol{H}(\boldsymbol{x},\omega) \\ \boldsymbol{B}(\boldsymbol{x},\omega) &= \mathcal{B}(\omega)\,\boldsymbol{E}(\boldsymbol{x},\omega) + \mathcal{D}(\omega)\,\boldsymbol{H}(\boldsymbol{x},\omega) \end{aligned}\right\}, \tag{44}$$

with the assumption of temporal invariance, spatial homogeneity, spatial locality, and isotropy. Furthermore, (44) apply only at sufficiently low frequencies [35].

Gyrators have been approximately realized using other circuit elements, but the Tellegen medium has never been successfully synthesized. Tellegen's own experiments failed [19, p. 96] Neither has the Tellegen medium been observed in nature. Hence, non-zero values of $\mathcal{B}(\omega)$ of actual materials are not known. A fairly elementary exercise shows that the recognizable existence of this medium is tied to that of irreducible magnetic sources [36, 37]. As the prospects of observing a magnetic monopole are rather remote [38, 39], for now it is appropriate to regard the Tellegen medium as chimerical.

5.3. Tellegen particle

Each particle in Tellegen's recipe is actually a uniaxial particle [40]. Because the recipe calls for the suspension to be homogeneous, the particles cannot be similarly oriented. However, if all particles were similarly oriented in free space, and the number density N_p of the particles is very small, the frequency-domain constitutive relations of the suspension at sufficiently low frequencies will be

$$\left.\begin{aligned} \boldsymbol{D}(\boldsymbol{x},\omega) &\simeq \varepsilon_0 \boldsymbol{E}(\boldsymbol{x},\omega) \\ &\quad + N_p(\boldsymbol{\pi}^{(ee)}(\omega)\cdot\boldsymbol{E}(\boldsymbol{x},\omega) + \boldsymbol{\pi}^{(eh)}(\omega)\cdot\boldsymbol{H}(\boldsymbol{x},\omega)) \\ \boldsymbol{B}(\boldsymbol{x},\omega) &\simeq \mu_0 \boldsymbol{H}(\boldsymbol{x},\omega) \\ &\quad + N_p(\boldsymbol{\pi}^{(he)}(\omega)\cdot\boldsymbol{E}(\boldsymbol{x},\omega) + \boldsymbol{\pi}^{(hh)}(\omega)\cdot\boldsymbol{H}(\boldsymbol{x},\omega)) \end{aligned}\right\}, \tag{45}$$

wherein $\boldsymbol{\pi}^{(ee)}$, etc., are the polarizability tensors of a Tellegen particle in free space.

A recent report [18] provides experimental evidence on the existence of $\boldsymbol{\pi}^{(eh)}$ for a Tellegen particle made by sticking a short copper wire to a ferrite sphere biased by a quasistatic magnetic field parallel to the wire. However, this work can not lead to any significant finding against the validity of the Post constraint for the following two reasons:

- Although a quantity proportional to the magnitude of $\mathrm{Trace}\big(\boldsymbol{\pi}^{(eh)}\big)$ was measured, a similar measurement of $\mathrm{Trace}\big(\boldsymbol{\pi}^{(he)}\big)$ was not undertaken; instead, the identity

  $$\mathrm{Trace}\ (\boldsymbol{\pi}^{(he)}(\omega)) = \mathrm{Trace}\ (\boldsymbol{\pi}^{(eh)}(\omega)) \tag{46}$$

  was assumed without testing. This deficiency in experimentation is similar to that for magnetoelectric materials mentioned in Section 5.1.
- The Post constraint is supposed to hold rigorously for linear electromagnetic response with respect to the total electromagnetic field, which is constituted jointly by the bias magnetic field as well as the time-harmonic electromagnetic field. As discussed by Chen [41], the ferrite is therefore a nonlinear material.

Incidentally, the biased-ferrite-metal-wire modality for Tellegen particles is likely to be very difficult to implement to realize the Tellegen medium of Section 5.2.

5.4. Summation of experimental evidence

On reviewing Sections 5.1–5.3, it becomes clear that experimental evidence against the validity of the Post constraint is incomplete and inconclusive, in addition

[3]) This and a large fraction of other published reports do not seem to recognize that $\mathcal{C}(\omega)$, etc., are complex-valued quantities, but treat them as real-valued quantities.

to being based either on the false premise of purely instantaneous response and/or derived from EH-electromagnetism.

6. Post constraint and free space

Although the Post constraint holds for modern electromagnetism, which has a microscopic basis in that matter is viewed as an assembly of charge-carriers in free space, before concluding this essay it is instructive to derive the constitutive equations of free space back from the macroscopic constitutive equations (21) and (22).

Let us begin with free space being spatiotemporally invariant and spatiotemporally local; then, $\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}, t; \boldsymbol{x}_h, t_h) = \tilde{\boldsymbol{\varepsilon}}_{fs}\delta(\boldsymbol{x}_h)\,\delta(t_h)$, etc., and (21) and (22) simplify to

$$\left.\begin{aligned}\tilde{\boldsymbol{D}}(\boldsymbol{x}, t) &= \tilde{\boldsymbol{\varepsilon}}_{fs} \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}, t) + \tilde{\boldsymbol{\xi}}_{fs} \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}, t)\\ \tilde{\boldsymbol{H}}(\boldsymbol{x}, t) &= \tilde{\boldsymbol{\zeta}}_{fs} \cdot \tilde{\boldsymbol{E}}(\boldsymbol{x}, t) + \tilde{\boldsymbol{\nu}}_{fs} \cdot \tilde{\boldsymbol{B}}(\boldsymbol{x}, t)\end{aligned}\right\}. \tag{47}$$

The free energy being a perfect differential, and because the constitutive relations (47) do not involve convolution integrals, it follows that [1, eq. 6.14]

$$\text{Transpose}\,(\tilde{\boldsymbol{\xi}}_{fs}) = -\tilde{\boldsymbol{\zeta}}_{fs}\,. \tag{48}$$

With the additional requirement of isotropy, we get

$$\left.\begin{aligned}\tilde{\boldsymbol{D}}(\boldsymbol{x}, t) &= \tilde{\varepsilon}_{fs}\,\tilde{\boldsymbol{E}}(\boldsymbol{x}, t) + \tilde{\xi}_{fs}\,\tilde{\boldsymbol{B}}(\boldsymbol{x}, t)\\ \tilde{\boldsymbol{H}}(\boldsymbol{x}, t) &= -\tilde{\xi}_{fs}\,\tilde{\boldsymbol{E}}(\boldsymbol{x}, t) + \tilde{\nu}_{fs}\,\tilde{\boldsymbol{B}}(\boldsymbol{x}, t)\end{aligned}\right\}. \tag{49}$$

The subsequent imposition of the Post constraint means that $\tilde{\xi}_{fs} = 0$, and the constitutive relations

$$\left.\begin{aligned}\tilde{\boldsymbol{D}}(\boldsymbol{x}, t) &= \tilde{\varepsilon}_{fs}\,\tilde{\boldsymbol{E}}(\boldsymbol{x}, t)\\ \tilde{\boldsymbol{H}}(\boldsymbol{x}, t) &= \tilde{\nu}_{fs}\,\tilde{\boldsymbol{B}}(\boldsymbol{x}, t)\end{aligned}\right\} \tag{50}$$

finally emerge. The values $\tilde{\varepsilon}_{fs} = \varepsilon_0$ and $\tilde{\nu}_{fs} = 1/\mu_0$ are used in SI [25]. Although Lorentz-reciprocity was not explicitly enforced for free space, it emerges naturally in this exercise [42]. Alternatively, it could have been enforced from the very beginning, and it would have led to $\tilde{\xi}_{fs} = 0$ [43].

7. Concluding remarks

Despite the fact that the mathematical forms of the macroscopic Maxwell postulates are identical in modern electromagnetism as well as in EH-electromagnetism, the two are very physically very different. Modern electromagnetism is held to be basic; hence, the answers to all fundamental questions must be decided within its framework. Thereafter, if necessary, its equations can be transformed into the frequency domain and *then* into those of EH-electromagnetism – and the resulting equations may be used to solve any problems that a researcher may be interested in. The reverse transition from EH-electromagnetism to modern electromagnetism can lead to false propositions.

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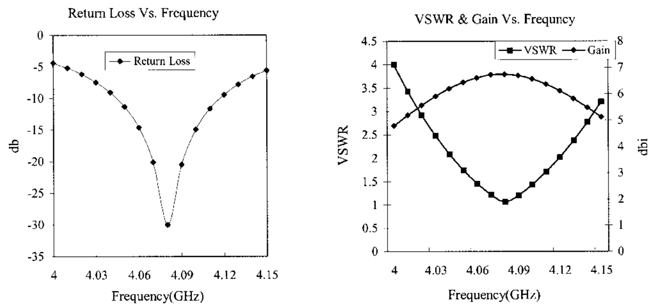

Figure 6 Return loss, VSWR, and gain for a compact microstrip patch antenna, with patch dimensions as in Fig. 5 (resonant frequency = 4.08 GHz)

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CONJUGATION SYMMETRY IN LINEAR ELECTROMAGNETISM IN EXTENSION OF MATERIALS WITH NEGATIVE REAL PERMITTIVITY AND PERMEABILITY SCALARS

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ABSTRACT: *If all space is occupied by a linear bianisotropic material—whether homogeneous or not—then the concurrent replacements of the permittivity and the impermeability tensors by the negatives of their respective complex conjugates and of the magneto-electric tensors by their respective complex conjugates (in the Boys–Post representation) imply the conjugation of both* $\underline{E}$ *and* $\underline{H}$*, in the absence of externally impressed sources. This conjugation symmetry in linear electromagnetism has observable consequences when the linear bianisotropic material occupies a bounded region.* © 2004 Wiley Periodicals, Inc. Microwave Opt Technol Lett 40: 160–161, 2004; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/mop.11315

Key words: *bianisotropy; conjugate invariance; conjugation symmetry; negative permittivity; negative permeability; reflection; transmission*

1. INTRODUCTION

The modest aim of this paper is to present a conjugation symmetry of frequency-domain electromagnetic fields in linear, nonhomogeneous, bianisotropic materials. This symmetry emerged as a generalization of a result obtained initially for linear, homogeneous, isotropic, dielectric-magnetic materials with negative real permittivity and permeability scalars [1–4]. Nominally, such a material possesses a relative permittivity scalar $\epsilon_r = \epsilon'_r + i\epsilon''_r$ and a relative permeability scalar $\mu_r = \mu'_r + i\mu''_r$, both dependent on the angular frequency ω, such that both $\epsilon'_r < 0$ and $\mu'_r < 0$ in some spectral regime; accordingly, the phase velocity vector and the time-averaged Poynting vector are oppositely directed in that spectral regime [4].* Originally conceived more than 35 years ago [1], these materials were artificially realized quite recently [2].

During an investigation of changes in frequency-domain electromagnetic fields when the transformation $\{\epsilon'_r \rightarrow -\epsilon'_r,\ \mu'_r \rightarrow -\mu'_r\}$ is effected on the isotropic dielectric-magnetic material occupying a source-free region, a more general conjugation symmetry in linear electromagnetism began to take shape. The following sections of this paper report the development of that symmetry.

2. CONJUGATE INVARIANCE OF MAXWELL POSTULATES

The frequency-domain Maxwell postulates may be written as

$$\left.\begin{aligned} \nabla\cdot\underline{D}(\underline{r},\omega) &= \rho_e(\underline{r},\omega)\\ \nabla\cdot\underline{B}(\underline{r},\omega) &= \rho_m(\underline{r},\omega)\\ \nabla\times\underline{E}(\underline{r},\omega) &= i\omega\underline{B}(\underline{r},\omega) - \underline{J}_m(\underline{r},\omega)\\ \nabla\times\underline{H}(\underline{r},\omega) &= -i\omega\underline{D}(\underline{r},\omega) + \underline{J}_e(\underline{r},\omega) \end{aligned}\right\}, \tag{1}$$

in the presence of externally impressed sources of the electric and magnetic types. These four postulates are collectively invariant with respect to the transformation

$$\{\underline{E} \rightarrow \underline{E}^*, \underline{H} \rightarrow \underline{H}^*, \underline{D} \rightarrow -\underline{D}^*, \underline{B} \rightarrow -\underline{B}^*, \underline{J}_e \rightarrow \underline{J}_e^*, \underline{J}_m \rightarrow \underline{J}_m^*, \rho_e \rightarrow -\rho_e^*, \rho_m \rightarrow -\rho_m^*\}, \tag{2}$$

where the asterisk denotes the complex conjugate. This statement of conjugate invariance may be verified by direct substitution of Eq. (2) in Eq. (1).

The conjugate invariance of the Maxwell postulates not only underlies a similarly invariant Beltrami form of electromagnetism [6], but also permits the existence of a conjugation symmetry in linear electromagnetism.

3. CONJUGATE INVARIANCE AND LINEAR BIANISOTROPY

There are two widely used sets of frequency-domain constitutive relations for linear bianisotropic materials [7]. Both are considered as follows:

3.1. Boys–Post Constitutive Relations

The Boys–Post constitutive relations of a linear, nonhomogeneous, bianisotropic material can be stated as

$$\left.\begin{aligned} \underline{D}(\underline{r},\omega) &= \underline{\underline{\epsilon}}(\underline{r},\omega)\cdot\underline{E}(\underline{r},\omega) + \underline{\underline{\alpha}}(\underline{r},\omega)\cdot\underline{B}(\underline{r},\omega)\\ \underline{H}(\underline{r},\omega) &= \underline{\underline{\beta}}(\underline{r},\omega)\cdot\underline{E}(\underline{r},\omega) + \underline{\underline{\chi}}(\underline{r},\omega)\cdot\underline{B}(\underline{r},\omega) \end{aligned}\right\}, \tag{3}$$

* The condition for the phase velocity and the time-averaged Poynting vectors to be oppositely directed is $(|\epsilon_r| - \epsilon'_r)(|\mu_r| - \mu'_r) > \epsilon''_r\mu''_r$, which permits—more generally—ϵ'_r and/or μ'_r to be negative [5]. An $\exp(-i\omega t)$ time-dependence having been assumed here, $\epsilon''_r > 0$ and $\mu''_r > 0$ at all $\omega > 0$ for all passive materials.

subject to the constraint Trace$\{\underline{\underline{\alpha}} - \underline{\underline{\beta}}\} \equiv 0$. Whereas $\underline{\underline{\epsilon}}$ is the permittivity tensor and $\underline{\underline{\chi}}$ is the impermeability tensor, both $\underline{\underline{\alpha}}$ and $\underline{\underline{\beta}}$ are magnetoelectric tensors.

The transformation

$$\{\underline{\underline{\epsilon}} \rightarrow -\underline{\underline{\epsilon}}^*, \underline{\underline{\chi}} \rightarrow -\underline{\underline{\chi}}^*, \underline{\underline{\alpha}} \rightarrow \underline{\underline{\alpha}}^*, \beta \rightarrow \underline{\underline{\beta}}^*\} \tag{4}$$

of constitutive tensors then entails the transformation

$$\{\underline{E} \rightarrow \underline{E}^*, \underline{H} \rightarrow \underline{H}^*, \underline{D} \rightarrow -\underline{D}^*, \underline{B} \rightarrow -\underline{B}^*\} \tag{5}$$

of the electromagnetic fields—in conformity with Eq. (2) expressing the conjugate invariance of the Maxwell postulates.

3.2. Tellegen Constitutive Relations

The Tellegen constitutive relations of a linear, nonhomogeneous, bianisotropic material can be stated as

$$\left.\begin{aligned} \underline{D}(\underline{r}, \omega) &= \underline{\underline{\hat{\epsilon}}}(\underline{r}, \omega) \cdot \underline{E}(\underline{r}, \omega) + \underline{\underline{\xi}}(\underline{r}, \omega) \cdot \underline{H}(\underline{r}, \omega) \\ \underline{B}(\underline{r}, \omega) &= \underline{\underline{\zeta}}(\underline{r}, \omega) \cdot \underline{E}(\underline{r}, \omega) + \underline{\underline{\mu}}(\underline{r}, \omega) \cdot \underline{H}(\underline{r}, \omega) \end{aligned}\right\}, \tag{6}$$

subject to the constraint Trace$\{\underline{\underline{\mu}}^{-1} \cdot (\underline{\underline{\xi}} + \underline{\underline{\zeta}})\} \equiv 0$. Here, $\underline{\underline{\hat{\epsilon}}}$ is the permittivity tensor, $\underline{\underline{\mu}}$ is the permeability tensor, and both $\underline{\underline{\xi}}$ and $\underline{\underline{\zeta}}$ are magnetoelectric tensors.

The transformation

$$\{\underline{\underline{\hat{\epsilon}}} \rightarrow -\underline{\underline{\hat{\epsilon}}}^*, \underline{\underline{\mu}} \rightarrow -\underline{\underline{\mu}}^*, \underline{\underline{\xi}} \rightarrow -\underline{\underline{\xi}}^*, \zeta \rightarrow -\underline{\underline{\zeta}}^*\} \tag{7}$$

of the constitutive tensors then also entails the field transformation (5), in conformity with the conjugate invariance of the Maxwell postulates.

Because the frequency-domain constitutive relations of the Boys–Post and the Tellegen types are intertranslatable, the constitutive-tensor transformations (4) and (7) are actually equivalent.

4. CONJUGATION SYMMETRY IN LINEAR ELECTROMAGNETISM

The deductions in section 3 permit the enunciation of the following conjugation symmetry. If all space were to be occupied by a linear bianisotropic material—whether homogeneous or not—then a change of the constitutive tensors as per Eq. (4) would imply the conjugation of both $\underline{E}(\underline{r}, \omega)$ and $\underline{H}(\underline{r}, \omega)$, in the absence of externally impressed sources. If such sources are present, then the conjugation symmetry is expressed jointly by Eqs. (2) and (4).

The effect of the transformation (4) would be observable even if the linear bianisotropic material were to be confined to a bounded region. For instance, imagine a slab of infinite transverse extent and uniform thickness, separating two vacuous half-spaces; the slab is made of a linear, isotropic, dielectric-magnetic material; and a linearly polarized, propagating, plane wave is incident on the slab. If the real parts of the permittivity and the permeability scalars of the slab were to change sign simultaneously, then the reflection and the transmission coefficients would have to be replaced by their respective complex conjugates [8]. The same conclusion holds if the slab were to be piecewise uniform in the thickness direction. The complex conjugation of reflection and transmission coefficients would essentially hold, even if the slab were to be bianisotropic and plane stratified, but a dependence on the polar angle of the incidence wavevector would also appear—as demonstrated elsewhere for chiral ferrosmectic slabs [9]. However, the conjugation of the reflection and the transmission coefficients would not hold on reversal of the signs of the real parts of the permittivity and permeability scalars of an isotropic, plane-stratified, dielectric-magnetic slab, if the incident plane wave were to evanesc.

The conclusions stated in the foregoing paragraph were obtained both by direct calculation and by application of the conjugation symmetry enunciated at the beginning of this section. The latter procedure is, of course, very simple; and it demonstrates the importance of the proffered symmetry in quickly determining the observable consequences of employing newly emerging materials (such as isotropic dielectric-magnetic materials with negative real permittivity and permeability scalars, and their anisotropic counterparts) for various applications.

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COMPARISON OF GRATING ANALYSIS TECHNIQUES FOR SG-DBR LASERS

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ABSTRACT: *Two techniques to locate the wavelengths of the main reflectivity peaks of a sampled grating used as part of a sampled-grating distributed Bragg reflector (SG-DBR) laser are presented. It is shown that the results are in good agreement and also consistent with the theoretical results.* © 2004 Wiley Periodicals, Inc. Microwave Opt Technol Lett 40: 161–164, 2004; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/mop.11316

Key words: *tunable lasers; sampled grating; distributed Bragg reflector*

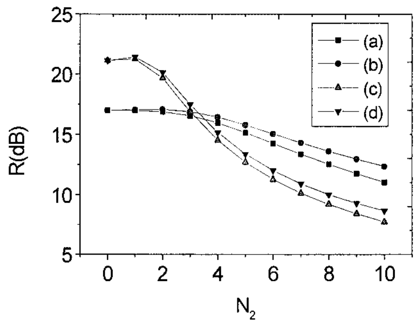

Figure 5 Switching contrast R for different cases of higher order dispersion. (a) $T_{c0} = 2.5T_{s0}$, $\mathrm{sgn}(\beta_{31}) = -\mathrm{sgn}(\beta_{32})$. (b) $T_{c0} = 2.5T_{s0}$, $\mathrm{sgn}(\beta_{31}) = \mathrm{sgn}(\beta_{32})$. (c) $T_{c0} = 3.5T_{s0}$, $\mathrm{sgn}(\beta_{31}) = -\mathrm{sgn}(\beta_{32})$. (d) $T_{c0} = 3.5T_{s0}$, $\mathrm{sgn}(\beta_{31}) = \mathrm{sgn}(\beta_{32})$

induce fewer interactions between the control and signal pulses, and so the switching contrast decreases slightly.

From the above analysis, we can see that higher order dispersion proportional to β_3 will cause NOLM performance degradation, such as a decrease of the switching contrast and an increase of the transmission signal pulse width. The value of N_2 defined by Eq. (3) can be a criterion for measuring the effects of HOD. Here, we choose the same parameters of $|\beta_{21}| = 0.028$ ps/km/nm and $T_{c0} = 0.5$ ps used in [5], which achieve 640 Gbit/s signal demultiplexing. In conventional dispersion-shifted fiber (DSF), β_3 has a typical value of $0.07\ \mathrm{ps/nm^2 \cdot km}$ [6]. So, according to Eq. (3), N_2 will be as high as 5 if DSF is used as the loop fiber, and thus poor performance of NOLM will be caused. Therefore, the conventional DSF fiber cannot achieve good performance in terabit/ second demultiplexing. In [5], dispersion-flattened fiber (DFF) is used instead of DSF since DFF has much less higher order dispersion ($\beta_3 = 0.0005\ \mathrm{ps/nm^2 \cdot km}$ in [7]).

4. CONCLUSION

In conclusion, when ultra-fast demultiplexing as high as hundreds of gigabits/second is operated, higher order dispersion must be considered. A large value of higher order dispersion will cause low switching contrast and signal pulse distortion and broadening. The criterion for measuring the effects of HOD is presented. Conventional DSF fiber is not suitable for terabit/second operation because of its relatively large β_3. Optical fiber with a small β_3, such as the DFF in [5], should be employed.

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BELTRAMI FIELD PHASORS ARE EIGENVECTORS OF 6 × 6 LINEAR CONSTITUTIVE DYADICS

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ABSTRACT: *When the constitutive parameters of a linear, homogeneous, bianisotropic medium are arranged in a certain way as a 6 × 6 dyadic, it is shown that the eigenvectors of that dyadic may yield admissible Beltrami field phasors whose wavenumbers are directly proportional to the corresponding eigenvalues.* © 2001 John Wiley & Sons, Inc. Microwave Opt Technol Lett 30: 127–128, 2001.

Key words: *Beltrami field; bianisotropic medium; eigenvalue problem*

1. INTRODUCTION

The intimate connection—as stated in the title of this communication—between the possible existence of time-harmonic Beltrami electromagnetic fields in a linear, homogeneous, bianisotropic medium and the constitutive properties of that medium is proved here. An $\exp(-i\omega t)$ time dependence is implicit throughout, where ω is the angular frequency and t is the time. Whereas 3-vectors (6-vectors) are in boldface and underlined, 3 × 3 dyadics (6 × 6 dyadics) are in boldface and underlined twice.

2. PRELIMINARIES

Fields that are either parallel or antiparallel to their own circulations are called Beltrami fields; thus, a Beltrami field $\underline{v}(\underline{x})$ satisfies the condition [1]

$$\nabla \times \underline{v}(\underline{x}) = \Omega \underline{v}(\underline{x}), \qquad \Omega \neq 0 \tag{1}$$

where $\underline{x}$ is the position vector and Ω is akin to a wavenumber. Electromagnetic field phasors in a linear, homogeneous, bianisotropic medium obey the constitutive relations

$$\left.\begin{aligned} \underline{D}(\underline{x}) &= \underline{\underline{\epsilon}} \cdot \underline{E}(\underline{x}) + \underline{\underline{\xi}} \cdot \underline{H}(\underline{x}) \\ \underline{B}(\underline{x}) &= \underline{\underline{\zeta}} \cdot \underline{E}(\underline{x}) + \underline{\underline{\mu}} \cdot \underline{H}(\underline{x}) \end{aligned}\right\} \tag{2}$$

where the 3 × 3 dyadics $\underline{\underline{\epsilon}}$, etc., are functions of the angular frequency. Equations (2) are subject to certain constraints [2]. Researchers working on complex media know well that Beltrami electromagnetic field phasors can be launched in all isotropic dielectric–magnetic media, and are actually indispensable when considering isotropic chiral materials [3]. Examples are also available as circularly polarized plane waves in magnetoplasmas [4], which are anisotropic materials.

3. ANALYSIS

The source-free, time-harmonic Maxwell curl postulates applicable to a medium described by (2) can be compactly written as

$$\underline{\underline{\mathbf{L}}}(\nabla)\cdot\underline{\mathbf{F}}(\underline{x}) = i\omega\underline{\underline{\mathbf{C}}}\cdot\underline{\mathbf{F}}(\underline{x}). \tag{3}$$

In this equation,

$$\underline{\mathbf{F}}(\underline{x}) = \begin{pmatrix}\underline{E}(\underline{x})\\ \underline{H}(\underline{x})\end{pmatrix}, \underline{\underline{\mathbf{L}}}(\nabla) = \begin{pmatrix}\nabla\times\underline{\underline{I}} & \underline{\underline{0}}\\ \underline{\underline{0}} & \nabla\times\underline{\underline{I}}\end{pmatrix} \tag{4}$$

with $\underline{\underline{I}}$ as the 3×3 identity dyadic and $\underline{\underline{0}}$ as the 3×3 null dyadic, while

$$\underline{\underline{\mathbf{C}}} = \begin{pmatrix}\underline{\underline{\zeta}} & \underline{\underline{\mu}}\\ -\underline{\underline{\epsilon}} & -\underline{\underline{\xi}}\end{pmatrix} \tag{5}$$

is the 6×6 constitutive dyadic. In consonance with (1), a Beltrami solution of (3) must obey the constraint

$$\underline{\underline{\mathbf{L}}}(\nabla)\cdot\underline{\mathbf{F}}(\underline{x}) = \Omega\underline{\mathbf{F}}(\underline{x}) \equiv i\omega\kappa\underline{\mathbf{F}}(\underline{x}) \tag{6}$$

where the scalar κ is an inverse speed. Substitution of (6) into (3) leads to the equation

$$\underline{\underline{\mathbf{C}}}\cdot\underline{\mathbf{F}}(\underline{x}) = \kappa\underline{\mathbf{F}}(\underline{x}). \tag{7}$$

Equation (7) is an eigenvalue equation, κ being an eigenvalue of the 6×6 constitutive dyadic as arranged in (5). For a specific κ, there will exist one or more distinct eigenvectors, an $\underline{x}$-dependent linear combination of which may satisfy (3). That combination will satisfy (6) too. In practical terms, it may be best to first obtain a nonzero eigenvalue of $\underline{\underline{\mathbf{C}}}$, then construct corresponding solutions of (6) as linear combinations of compatible poloidal and toroidal fields [3, 5], and finally check if any one of those solutions of (6) also solves (3). If yes, then $\underline{D}(\underline{x}) = -\kappa\underline{H}(\underline{x})$ and $\underline{B}(\underline{x}) = \kappa\underline{E}(\underline{x})$ are easy to obtain thereafter.

Thus, a time-harmonic Beltrami electromagnetic field *may* be possible in any linear, homogeneous, bianisotropic medium for every nonzero eigenvalue of $\underline{\underline{\mathbf{C}}}$. To the author's knowledge, this *intimate* relationship between Beltrami field phasors and the constitutive properties of a linear, homogeneous, bianisotropic medium has never been reported previously. The implications of this relationship are presently under investigation.

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QUADRIFILAR HELIX ANTENNA WITH PARASITIC HELICAL STRIPS

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ABSTRACT: *An omnidirectional quadrifilar helix antenna (QHA) with parasitic helical strips for circular polarization (CP) is proposed and experimentally investigated for handheld mobile terminals. This antenna is small in size, low cost, and lightweight. The mutual coupling effect between the grounded helical strips and the feeding helical arms provides a good impedance match, a smaller axial ratio, and wider hemispherical coverage. The impedance bandwidth corresponding to SWR < 2 and bandwidth with respect to the axial ratio < 3 dB are found to be 39% and 160 MHz at 3.64 GHz, respectively.* © 2001 John Wiley & Sons, Inc. Microwave Opt Technol Lett 30: 128–130, 2001.

Key words: *quadrifilar helix antenna (QHA); parasitic helical strips; circular polarization (CP)*

1. INTRODUCTION

Circularly polarized (CP) antennas find wide applications in mobile satellite communications and direct broadcasting satellite systems due to their insensitivity to ionospheric polarization rotation. In recent years, the quadrifilar helix antenna (QHA) [1] has been widely used in satellite telecommunication systems, such as ground receivers for the global positioning system (GPS) and ground earth terminals (handheld) in low-elevation-orbit (LEO) satellite communications. In mobile satellite communications systems, an omnidirectional beam of a CP antenna is preferable for vehicle antennas because no satellite tracking system in the azimuthal direction is needed. The elevation angle of the beam direction can also be adjusted for vehicles at different latitudes and areas [2]. The QHA differs from a conventional helical antenna in that it has four windings of partial or multiple turns [3]. A conventional QHA suffers from the problem of narrow bandwidth and an ill-matched input impedance. This is due to its operation under resonant modes instead of traveling modes. In this letter, we propose a QHA with a second parasitic QHA made of copper strips. It is found that, with the introduction of this parasitic QHA, the performance of the QHA in terms of matching efficiency, bandwidth with circular polarization, and radiation coverage can be significantly improved. The proposed QHA will be studied experimentally.

2. ANTENNA CONFIGURATION

The QHA is shown in Figure 1. Four equal-length copper wires of diameter δ are wound as four helixes on a cylindrical core of diameter d. The lengths of the copper wires are

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Conditions for Circularly Polarized Plane Wave Propagation in a Linear Bianisotropic Medium

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Derived here are four conditions that must be satisfied by the constitutive parameters of a linear, homogeneous, bianisotropic medium for it to support the propagation of a circularly polarized plane wave along a fixed axis.

Keywords bianisotropic medium, circular polarization, plane wave

Introduction

Suppose a circularly polarized plane wave is required to be propagated in a particular linear, homogeneous medium in a direction parallel to the x_3 axis of a fixed coordinate system in which the position vector $\underline{x} = \sum_{n=1}^{3} x_n \underline{\hat{u}}_n$ and $\underline{\hat{u}}_{1,2,3}$ are cartesian unit vectors. The requirement is easy to fulfill if the medium has to be isotropic dielectric/magnetic (Krauss, 1984). It can also be easily fulfilled if the medium has to be isotropic chiral (Lakhtakia, 1994). It can be satisfied even in a magnetoplasma, but only if the bias magnetostatic field is also directed parallel to the x_3 axis (Chen, 1983). What about in some other bianisotropic medium?

The objective of this communication is to derive *four* conditions that must be satisfied by the constitutive parameters of a linear, homogeneous medium for it to support the propagation of a circularly polarized plane wave along the x_3 axis. An $\exp(-i\omega t)$ time-dependence is implicit throughout; vectors are underlined once and dyadics are underlined twice.

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Analysis

Electromagnetic field phasors in a linear, homogeneous, bianisotropic medium obey the constitutive relations

$$\begin{aligned}\underline{D}(\underline{x}) &= \underline{\underline{\epsilon}} \bullet \underline{E}(\underline{x}) + \underline{\underline{\xi}} \bullet \underline{H}(\underline{x}),\\ \underline{B}(\underline{x}) &= \underline{\underline{\zeta}} \bullet \underline{E}(\underline{x}) + \underline{\underline{\mu}} \bullet \underline{H}(\underline{x}),\end{aligned} \tag{1}$$

wherein the 3×3 constitutive dyadics $\underline{\underline{\epsilon}}$, etc., are implicit functions of the angular frequency and are subject to certain constraints (Weiglhofer, 1998). In view of (1) and the time-harmonic Maxwell curl postulates, the electromagnetic field phasors associated with an x_3-traveling plane wave must be the solution of the two equations (Weiglhofer, 1995)

$$\underline{\hat{u}}_1 \frac{d}{dx_3} E_2(x_3) - \underline{\hat{u}}_2 \frac{d}{dx_3} E_1(x_3) = -i\omega \left[\underline{\underline{\zeta}} \bullet \underline{E}(x_3) + \underline{\underline{\mu}} \bullet \underline{H}(x_3)\right], \tag{2}$$

$$\underline{\hat{u}}_1 \frac{d}{dx_3} H_2(x_3) - \underline{\hat{u}}_2 \frac{d}{dx_3} H_1(x_3) = i\omega \left[\underline{\underline{\epsilon}} \bullet \underline{E}(x_3) + \underline{\underline{\xi}} \bullet \underline{H}(x_3)\right], \tag{3}$$

where $E_n \equiv \underline{\hat{u}}_n \bullet \underline{E}$ $(1 \le n \le 3)$, etc. Any circularly polarized plane wave must be transverse; furthermore, it can be either left- or right-circularly polarized. Therefore, the following characteristics are required of the solution of (2) and (3):

$$\begin{aligned}E_2(x_3) &= \pm i E_1(x_3), \quad E_3(x_3) = 0,\\ H_2(x_3) &= \pm i H_1(x_3), \quad H_3(x_3) = 0.\end{aligned} \tag{4}$$

Either the upper signs or the lower signs must be consistently chosen in (4) and the following equations, depending on the helicity desired of the plane wave.

On taking the inner products of both sides of (2) and (3) with $\underline{\hat{u}}_3$, and after making use of the requirements (4), we obtain the matrix equation

$$\begin{bmatrix} \zeta_{31} \pm i\zeta_{32} & \mu_{31} \pm i\mu_{32} \\ \epsilon_{31} \pm i\epsilon_{32} & \xi_{31} \pm i\xi_{32} \end{bmatrix} \begin{bmatrix} E_1(x_3) \\ H_1(x_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5}$$

where $\epsilon_{mn} = \underline{\hat{u}}_m \bullet \underline{\underline{\epsilon}} \bullet \underline{\hat{u}}_n$, etc. This equation has a nontrivial solution only if the 2×2 matrix on its left side is singular, which yields the *first* of the desired four conditions as follows:

$$(\zeta_{31} \pm i\zeta_{32})(\xi_{31} \pm i\xi_{32}) = (\epsilon_{31} \pm i\epsilon_{32})(\mu_{31} \pm i\mu_{32}). \tag{6}$$

Next, we take the inner products of both sides of (2) with $\underline{\hat{u}}_1$ and $\underline{\hat{u}}_2$, respectively, and compare the results to obtain

$$[(\zeta_{11} - \zeta_{22}) \pm i(\zeta_{12} + \zeta_{21})]\; E_1(x_3) + [(\mu_{11} - \mu_{22}) \pm i(\mu_{12} + \mu_{21})]\; H_1(x_3) = 0. \tag{7}$$

Similar manipulations with (3) give the analogous equation

$$[(\epsilon_{11} - \epsilon_{22}) \pm i(\epsilon_{12} + \epsilon_{21})]\; E_1(x_3) + [(\xi_{11} - \xi_{22}) \pm i(\xi_{12} + \xi_{21})]\; H_1(x_3) = 0. \tag{8}$$

Equations (7) and (8) have a nontrivial solution if and only if the following condition holds true:

$$
\begin{aligned}
&[(\zeta_{11}-\zeta_{22}) \pm i(\zeta_{12}+\zeta_{21})][(\xi_{11}-\xi_{22}) \pm i(\xi_{12}+\xi_{21})] \\
&\quad = [(\epsilon_{11}-\epsilon_{22}) \pm i(\epsilon_{12}+\epsilon_{21})][(\mu_{11}-\mu_{22}) \pm i(\mu_{12}+\mu_{21})]\,.
\end{aligned} \tag{9}
$$

This is the *second* of the four conditions desired.

Now we can determine all the other characteristics of the plane wave of interest. Taking the inner products of both sides of (2) and (3) with $\underline{\hat{u}}_2$, and insisting on the requirements (4), we get the matrix differential equation

$$
\frac{d}{dx_3}\begin{bmatrix} E_1(x_3) \\ H_1(x_3) \end{bmatrix} = i\omega \begin{bmatrix} \zeta_{21} \pm i\zeta_{22} & \mu_{21} \pm i\mu_{22} \\ -(\epsilon_{21} \pm i\epsilon_{22}) & -(\xi_{21} \pm i\xi_{22}) \end{bmatrix} \begin{bmatrix} E_1(x_3) \\ H_1(x_3) \end{bmatrix}. \tag{10}
$$

After making the reasonable assumption that the two eigenvectors of the matrix on the right side of (10) are distinct, and ignoring the anomalous possibility of Voigt wave propagation (Khapalyuk, 1962), the solution of this equation is found using a standard technique (Hochstadt, 1975) as

$$
\begin{aligned}
E_1(x_3) &= A_a \exp(ik_a x_3) + A_b \exp(ik_b x_3), \\
H_1(x_3) &= \alpha_a A_a \exp(ik_a x_3) + \alpha_b A_b \exp(ik_b x_3),
\end{aligned} \tag{11}
$$

where A_a and A_b are complex-valued amplitudes, the two admittances are given by

$$
\begin{aligned}
\alpha_a &= \frac{(k_a/\omega) - (\zeta_{21} \pm i\zeta_{22})}{\mu_{21} \pm i\mu_{22}} = -\frac{\epsilon_{21} \pm i\epsilon_{22}}{(k_a/\omega) + \xi_{21} \pm i\xi_{22}}, \\
\alpha_b &= \frac{(k_b/\omega) - (\zeta_{21} \pm i\zeta_{22})}{\mu_{21} \pm i\mu_{22}} = -\frac{\epsilon_{21} \pm i\epsilon_{22}}{(k_b/\omega) + \xi_{21} \pm i\xi_{22}},
\end{aligned} \tag{12}
$$

and the corresponding wavenumbers are

$$
\begin{aligned}
k_a = \frac{\omega}{2}\{&[\zeta_{21} \pm i\zeta_{22} - (\xi_{21} \pm i\xi_{22})] \\
&+ [(\zeta_{21} \pm i\zeta_{22} + \xi_{21} \pm i\xi_{22})^2 - 4(\epsilon_{21} \pm i\epsilon_{22})(\mu_{21} \pm i\mu_{22})]^{1/2}\}, \\
k_b = \frac{\omega}{2}\{&[\zeta_{21} \pm i\zeta_{22} - (\xi_{21} \pm i\xi_{22})] \\
&- [(\zeta_{21} \pm i\zeta_{22} + \xi_{21} \pm i\xi_{22})^2 - 4(\epsilon_{21} \pm i\epsilon_{22})(\mu_{21} \pm i\mu_{22})]^{1/2}\}.
\end{aligned} \tag{13}
$$

At this point, the ratio $E_1(x_3)/H_1(x_3)$ can have three different values: the first from (5), the second from (7) and (8), and the third from (11). Reconciliation of these different values provides the *third* and the *fourth* conditions:

$$
\left.
\begin{aligned}
&[(\zeta_{11}-\zeta_{22}) \pm i(\zeta_{12}+\zeta_{21})] + \alpha_j\,[(\mu_{11}-\mu_{22}) \pm i(\mu_{12}+\mu_{21})] = 0, \\
&(\zeta_{31} \pm i\zeta_{32}) + \alpha_j\,(\mu_{31} \pm i\mu_{32}) = 0,
\end{aligned}
\right\} \quad j = a,\, b. \tag{14}
$$

Thus, we see that a circularly polarized plane wave can travel, in a particular bianisotropic medium, parallel to the x_3 axis of a fixed coordinate system, provided the constitutive dyadics of that medium satisfy the four conditions (6), (9), and (14). The field phasors associated with such a plane wave are given by

$$\begin{aligned} \underline{E}(x_3) &= A_j\, (\hat{\underline{u}}_1 \pm i\hat{\underline{u}}_2)\, \exp(ik_j x_3), \\ \underline{H}(x_3) &= \alpha_j\, A_j\, (\hat{\underline{u}}_1 \pm i\hat{\underline{u}}_2)\, \exp(ik_j x_3), \end{aligned} \qquad j = a,\, b. \tag{15}$$

Of course, the real part of the wavenumber k_j should not be null-valued for propagation, with or without attenuation, to occur.

Conclusion

All four conditions (6), (9), and (14) are easily satisfied by any medium with direction-independent properties. This is because both (6) and (9) reduce to trivialities as they involve either terms such as ϵ_{mn}, $m \neq n$, or differences such as $\epsilon_{nn} - \epsilon_{mm}$. Hence, circularly polarized planewaves can propagate in any direction in isotropic dielectric/magnetic (Krauss, 1984) and isotropic chiral mediums (Lakhtakia, 1994).

Another example is furnished by Faraday chiral mediums, e.g., magnetoplasmas (Chen, 1983), chiroplasmas, and chiroferrites (Weiglhofer & Lakhtakia, 1998). All four constitutive dyadics of a Faraday chiral medium are of the form

$$\underline{\underline{c}} = c_\perp\, (\underline{\underline{I}} - \hat{\underline{u}}_c\hat{\underline{u}}_c) + c_\parallel\, \hat{\underline{u}}_c\hat{\underline{u}}_c + ic_g\hat{\underline{u}}_c \times \underline{\underline{I}}, \qquad \underline{\underline{c}} = \underline{\underline{\epsilon}},\, \underline{\underline{\mu}},\, \underline{\underline{\zeta}},\, \underline{\underline{\xi}}, \tag{16}$$

where $\underline{\underline{I}}$ is the identity dyadic and $\hat{\underline{u}}_c$ is a unit vector parallel to the distinguished axis of the medium. As all four conditions (6), (9), and (14) are easily satisfied when $\hat{\underline{u}}_c = \hat{\underline{u}}_3$, circularly polarized propagation must occur parallel to its distinguished axis in a Faraday chiral medium.

Subject to the simultaneous satisfaction of the four conditions (6), (9), and (14), it is easy to see from the foregoing analysis that (i) the electromagnetic field phasors of the circularly polarized plane wave are an eigenvector of the 6× 6 constitutive dyadic

$$\underline{\underline{\mathsf{C}}} = \begin{Bmatrix} \underline{\underline{\zeta}} & \underline{\underline{\mu}} \\ -\underline{\underline{\epsilon}} & -\underline{\underline{\xi}} \end{Bmatrix}, \tag{17}$$

and (ii) the particular wavenumber is proportional to the corresponding eigenvalue of $\underline{\underline{\mathsf{C}}}$. Circularly polarized plane waves are Beltrami fields. Thus, this communication illustrates the recent discovery that Beltrami field phasors are eigenvectors of 6× 6 constitutive dyadics (Lakhtakia, 2001).

To conclude, as the dimensionality of the constitutive parameter space spanned by the four dyadics in the relations (1) is 70 (Michel, 2000), many more examples of the satisfaction of (6), (9), and (14) are expected to come to light.

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TABLE 2(b) Data Routing of One to Multiple (in Different Group)

	Input Encoder	Transducer		Output Decoder	
Prime code	$P3C0,0$	$P3C0,0$	$P3C0,0$	$P3C0,0$	$P3C2,0$
Routing path	In1	$A2$	$A3$	$A4$	Out1
Prime code		$P3C1,0$	$P3C0,2$	$P3C1,0$	$P3C0,2$
Routing path		$B2$	$E3$	$D4$	Out4
Prime code		$P3C2,0$	$P3C0,1$	$P3C2,0$	$P3C0,1$
Routing path		$C2$	$I3$	$G4$	Out7

TABLE 3 Data Routing of Broadcast

	Input Encoder	Transducer		Output Decoder	
Prime code	$P3C0,0$	$P3C0,0$	$P3C0,0$	$P3C0,0$	$P3C0,0$
Routing path	In1	$A2$	$A3$	$A4$	Out1
Prime code		$P3C0,0$	$P3C0,0$	$P3C2,2$	$P3C1,0$
Routing path		$A2$	$A3$	$B4$	Out2
Prime code		$P3C0,0$	$P3C0,0$	$P3C1,2$	$P3C2,0$
Routing path		$A2$	$A3$	$C4$	Out3
Prime code		$P3C1,0$	$P3C0,2$	$P3C1,0$	$P3C0,2$
Routing path		$B2$	$E3$	$D4$	Out4
Prime code		$P3C1,0$	$P3C0,2$	$P3C0,2$	$P3C1,1$
Routing path		$B2$	$E3$	$E4$	Out5
Prime code		$P3C1,0$	$P3C0,2$	$P3C2,1$	$P3C2,2$
Routing path		$B2$	$E3$	$F4$	Out6
Prime code		$P3C2,0$	$P3C0,1$	$P3C2,0$	$P3C0,1$
Routing path		$C2$	$I3$	$G4$	Out7
Prime code		$P3C2,0$	$P3C0,1$	$P3C1,1$	$P3C1,2$
Routing path		$C2$	$I3$	$H4$	Out8
Prime code		$P3C2,0$	$P3C0,1$	$P3C0,1$	$P3C2,1$
Routing path		$C2$	$I3$	$I4$	Out9

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A REPRESENTATION THEOREM INVOLVING FRACTIONAL DERIVATIVES FOR LINEAR HOMOGENEOUS CHIRAL MEDIA

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ABSTRACT: *A dyadic differential operator that commutes with the curl dyadic can be used to obtain new solutions of the Faraday and the Ampere–Maxwell equations in linear, homogeneous chiral media. Conditional extension of this representation theorem to bianisotropic media is also possible. An admissible operator may involve fractional derivatives.* © 2001 John Wiley & Sons, Inc. Microwave Opt Technol Lett 28: 385–386, 2001.

Key words: *bianisotropic media; chiral media; field representation; fractional derivatives*

This communication is motivated by two recent papers on the fractional dual solutions of the time-harmonic Maxwell postulates in vacuum [1] and isotropic chiral media [2].

Consider the time-harmonic Faraday and Amperé–Maxwell equations

$$\underline{\underline{L}} \cdot \underline{E} = i\omega(\mu \underline{H} - \xi \underline{E}) - \underline{K} \tag{1}$$

$$\underline{\underline{L}} \cdot \underline{H} = -i\omega(\epsilon \underline{E} + \xi \underline{H}) + \underline{J} \tag{2}$$

where $\underline{\underline{L}} \equiv \nabla \times \underline{\underline{I}}$ is the curl dyadic with $\underline{\underline{I}}$ as the identity dyadic, ω is the angular frequency, $\{\epsilon, \mu, \xi\}$ are the (frequency-dependent) constitutive parameters of a linear, homogeneous chiral medium, while $\{\underline{E}, \underline{H}\}$ are the field phasors and $\{\underline{J}, \underline{K}\}$ are the source current density phasors. Consider, also, a dyadic differential operator $\underline{\underline{M}}$ which commutes with $\underline{\underline{L}}$, i.e., $\underline{\underline{M}} \cdot \underline{\underline{L}} \equiv \underline{\underline{L}} \cdot \underline{\underline{M}}$. Applying $\underline{\underline{M}} \cdot$ from the left to both sides of Eqs. (1) and (2), we quickly arrive at the following *representation* theorem.

Theorem 1. If the sources $\{\underline{J}, \underline{K}\}$ give rise to the solutions $\{\underline{E}, \underline{H}\}$ of the time-harmonic Faraday and Amperé–Maxwell equations in a linear, homogeneous chiral medium, then $\{\underline{\underline{M}} \cdot \underline{J}, \underline{\underline{M}} \cdot \underline{K}\}$ must be the sources of the solutions $\{\underline{\underline{M}} \cdot \underline{E}, \underline{\underline{M}} \cdot \underline{H}\}$, provided $\underline{\underline{M}}$ and $\underline{\underline{L}}$ commute with each other.

A corollary is as follows.

Corollary 2. If $\{\underline{E}, \underline{H}\}$ are the solutions of the time-harmonic Faraday and Amperé–Maxwell equations in a linear, homogeneous chiral medium when sources are absent, then so are $\{\underline{\underline{M}} \cdot \underline{E}, \underline{\underline{M}} \cdot \underline{H}\}$, provided $\underline{\underline{M}} \cdot \underline{\underline{L}} \equiv \underline{\underline{L}} \cdot \underline{\underline{M}}$.

A simple admissible example of $\underline{\underline{M}}$ is the Laplacian operator; see Shanker [3] for an elegant use of its commutation with $\underline{\underline{L}}$ in determining the dyadic Green functions for Beltrami fields. More generally,

$$\underline{\underline{M}} = \left(\sum_j a_j \frac{\partial^{\alpha_j}}{\partial x^{\alpha_j}} \frac{\partial^{\beta_j}}{\partial y^{\beta_j}} \frac{\partial^{\gamma_j}}{\partial z^{\gamma_j}} \right) \underline{\underline{I}}, \qquad |a_j|, \alpha_j, \beta_j, \gamma_j \in [0, \infty) \tag{3}$$

is admissible. Like other operators involving fractional derivatives [4], it may be interpreted through the spatial Fourier transform.

References [1] and [2] illustrated *ex ante facto* Corollary 2 for $\underline{\underline{M}}$ as a fractional curl operator:

$$\underline{\underline{M}} = \underline{\underline{L}}^{\alpha}, \qquad \alpha \in [0, \infty). \tag{4}$$

The same choice of $\underline{\underline{M}}$ is admissible for Theorem 1 as well.

The foregoing apply, of course, to vacuum as well as isotropic dielectric/magnetic/dielectric–magnetic media. More significantly, however, both Theorem 1 and Corollary 2 can be extended to homogeneous bianisotropic media—with the frequency-domain constitutive relations

$$\left.\begin{aligned} \underline{D} &= \underline{\underline{\epsilon}} \cdot \underline{E} + \underline{\underline{\xi}} \cdot \underline{H} \\ \underline{B} &= \underline{\underline{\mu}} \cdot \underline{H} + \underline{\underline{\zeta}} \cdot \underline{E} \end{aligned}\right\} \tag{5}$$

provided $\underline{\underline{M}}$ also commutes with the constitutive dyadics $\underline{\underline{\epsilon}}$, $\underline{\underline{\mu}}$, $\underline{\underline{\xi}}$, and $\underline{\underline{\zeta}}$. For instance, the $\underline{\underline{M}}$ defined in Eq. (3) is admissible for a bianisotropic medium if all four of its constitutive dyadics are diagonal.

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A HIGH-PERFORMANCE ELECTROMAGNETICALLY COUPLED SHORTED CIRCULAR MICROSTRIP ANTENNA

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ABSTRACT: *The high-performance characteristics of an annular-ring-coupled shorted circular microstrip antenna are presented. This new geometry offers larger impedance bandwidth with a good return loss characteristic, a higher gain and radiation efficiency, and most importantly, a pattern characteristic with lower cross-polarization levels in the H- as well as E-planes. This new structure is investigated both theoretically and experimentally.* © 2001 John Wiley & Sons, Inc. Microwave Opt Technol Lett 28: 386–388, 2001.

Key words: *shorted microstrip antennas; broadband microstrip antennas; electromagnetically coupled stacked microstrip antennas; compact microstrip antennas*

1. INTRODUCTION

Small size, light weight, low profile, broad bandwidth, and proper polarization are the fundamental demands in antenna design for wireless communication systems. Microstrip antennas, for a long time, have been attractive choices for such applications. To reduce the size of the microstrip patch antenna, a large number of techniques have been proposed, such as shorting posts at strategic locations [1, 2], using high-dielectric-constant substrates, and removing nonradiating edges. Among these techniques, the use of shorting posts at strategic locations reduces the size of the patch antenna effectively. But the shorting post disturbs the radiation characteristics, such as a shift in the pattern, high cross-polarization levels, a dip in the *E*-plane pattern, and low radiation efficiency, and consequently reduces the gain of the patch antenna. To overcome these problems, and also to enhance the bandwidth, several techniques have been proposed [4, 5].

In this paper, to improve the cross-polarization levels, the shape of the patterns, the radiation efficiency, and the gain and bandwidth of the shorted circular microstrip antenna, a stacked annular-ring-coupled shorted circular microstrip antenna is presented. This new structure offers a significant reduction in the cross-polarized fields of a shorted patch, and at the same time is found to exhibit a larger impedance bandwidth with a higher gain as well as radiation efficiency. A common technique of enhancing the bandwidth by parasitically coupling another microstrip radiator to the driven element of the shorted patch is employed. In lieu of a coplanar version of parasitic coupling, a concentric vertically stacked electromagnetically coupled (EMC) geometry is utilized. Thus, the new configuration presented here consists of an annular-ring parasitic radiator, stacked and EM coupled to a concentric shorted circular-driven patch. The bottom circular-driven patch element with a shorting post reduces the size of the antenna, and also excites the dominant TM_{11} mode on the stacked annular ring, thereby utilizing the resonance of both radiators to significantly improve the bandwidth. This EMC

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Short Note

Conditions for Voigt wave propagation in linear, homogeneous, dielectric mediums

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Abstract: Two explicit conditions are derived for the permittivity dyadic of a linear, homogeneous, dielectric medium, in order that Voigt wave propagation may occur therein at the frequency of interest.

Key words: Plane waves – non-orthogonality – Voigt waves

1. Introduction

That two plane waves with mutually orthogonal polarizations, in general, can travel along any particular direction in a linear, homogeneous, dielectric medium is common knowledge in the electromagnetics/optics community [1]. What is not widely known is that there are instances when the two plane waves are non-orthogonal, a fact evidently first noticed by Voigt [2, 3] and later studied experimentally [4] as well as theoretically [5] by Pancharatnam. Furthermore, the two plane waves *then* coalesce into a single plane wave with an amplitude that varies linearly with the propagation distance. This composite wave is the so-called Voigt wave, and a trenchant commentary thereon by Khapalyuk [6] is highly recommended; see also Fedorov and Goncharenko [7].

Although Voigt waves have been examined in all classes of crystals [8], explicit conditions for their occurrence in all possible types of dielectric mediums do not appear to be available. This lack has begun to assume a certain significance with the increasing attention being paid nowadays to the fabrication, characterization and utilization of complex composite materials [9, 10]. In particular, composite materials comprising ellipsoidal inclusions of various materials randomly embedded in different host materials present a rich palette of macroscopic constitutive properties that are just beginning to be understood [11, 12, 13]. Comprehensive characterization schemes for these materials must take the possibility of Voigt wave propagation into account [14].

Accordingly, in this communication, we present two explicit conditions that must be satisfied by the permittivity dyadic of a linear, homogeneous, dielectric medium for Voigt wave propagation to occur therein at the frequency of interest.

2. Analysis

Consider fields with an inherent $\exp(-i\omega t)$ time-dependence, where ω is the angular frequency and t denotes the time. The constitutive relations (in the frequency domain) of a homogeneous, linear, dielectric medium are given by

$$\left.\begin{array}{l} \underline{D}(\underline{x}) = \underline{\underline{\varepsilon}} \cdot \underline{E}(\underline{x}) \\ \underline{B}(\underline{x}) = \mu_0 \underline{H}(\underline{x}) \end{array}\right\} . \tag{1}$$

Here, $= \sum_{m=1}^{3} x_m \underline{u}_m$ is the position vector with $\underline{u}_m$, $1 \le m \le 3$, being the three unit cartesian vectors; $\underline{E}(\underline{x}) = \sum_{m=1}^{3} E_m(\underline{x})\, \underline{u}_m$, etc., are the electromagnetic field phasors; $\underline{\underline{\varepsilon}} = \sum_{m=1}^{3} \sum_{n=1}^{3} \varepsilon_{mn} \underline{u}_m \underline{u}_n$ is the frequency-dependent permittivity dyadic of the chosen medium; while $\mu_0 = 4\pi \times 10^{-7}$ H/m is the permeability of vacuum.

Without loss of generality, consider propagation parallel to the unit vector $\underline{u}_3$, i.e., $\underline{E}(\underline{x}) = \underline{E} \exp(ikx_3)$, etc., where k denotes the wavenumber. The applicable source-free Maxwell curl postulates may then be writ-

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ten as

$$\left.\begin{aligned} k\underline{u}_3 \times \underline{\tilde{E}} - \omega \underline{\tilde{B}} = \underline{0} \\ k\underline{u}_3 \times \underline{\tilde{H}} + \omega \underline{\tilde{D}} = \underline{0} \end{aligned}\right\} . \tag{2}$$

Six algebraic equations are obtained on substituting (1) into (2). First, $\tilde{E}_3$ and $\tilde{H}_3$ are eliminated therefrom; and then $\tilde{H}_1$ and $\tilde{H}_2$ as well. The resulting two equations are best expressed together in matrix notation as

$$\frac{\omega^2 \mu_0}{\varepsilon_{33}} \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{bmatrix} = k^2 \begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{bmatrix}, \tag{3}$$

where

$$\left.\begin{aligned} \delta_{11} = \varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}\varepsilon_{31} \quad \delta_{12} = \varepsilon_{12}\varepsilon_{33} - \varepsilon_{13}\varepsilon_{32} \\ \delta_{21} = \varepsilon_{21}\varepsilon_{33} - \varepsilon_{23}\varepsilon_{31} \quad \delta_{22} = \varepsilon_{22}\varepsilon_{33} - \varepsilon_{23}\varepsilon_{32} \end{aligned}\right\} . \tag{4}$$

The matrix on the left side of (3) can have at most two independent eigenvectors, which is the normal scenario. The solution of (2) may be then written essentially as

$$\begin{bmatrix} E_1(x_3) \\ E_2(x_3) \end{bmatrix} = C_a \begin{bmatrix} \tilde{E}_{1a} \\ \tilde{E}_{2a} \end{bmatrix} \exp\left(\mathrm{i}k_a x_3\right) + C_b \begin{bmatrix} \tilde{E}_{1b} \\ \tilde{E}_{2b} \end{bmatrix} \exp\left(\mathrm{i}k_b x_3\right), \tag{5}$$

where C_a and C_b are amplitude coefficients, while $[\tilde{E}_{1a}, \tilde{E}_{2a}]^T$ and $[\tilde{E}_{1b}, \tilde{E}_{2b}]^T$ are the eigenvectors,[1] corresponding to the wavenumbers

$$k_a = \omega \left[\frac{\mu_0}{2\varepsilon_{33}} \left\{ (\delta_{11} + \delta_{22}) + \left[(\delta_{11} - \delta_{22})^2 + 4\delta_{12}\delta_{21} \right]^{1/2} \right\} \right]^{1/2} \tag{6}$$

and

$$k_b = \omega \left[\frac{\mu_0}{2\varepsilon_{33}} \left\{ (\delta_{11} + \delta_{22}) - \left[(\delta_{11} - \delta_{22})^2 + 4\delta_{12}\delta_{21} \right]^{1/2} \right\} \right]^{1/2}, \tag{7}$$

respectively. Both wavenumbers are the same (i.e., $k_a = k_b$) in isotropic mediums, but the eigenvectors are still independent. In anisotropic mediums, each of the two independent eigenvectors generally corresponds to a different wavenumber.

Voigt wave propagation occurs if and only if the matrix in (3) has just one independent eigenvector. Then, its two eigenvalues must be the same, and it must not be a scalar matrix. Mathematically, the two conditions for Voigt wave propagation to occur along the $+x_3$ axis are as follows:

A. $(\delta_{11} - \delta_{22})^2 + 4\delta_{12}\delta_{21} = 0$, and

B. at least one of δ_{12} and δ_{21} is nonzero.

[1] The superscript ^{T} denotes the transpose.

Provided both conditions hold, the essential solution of (2) may be stated as

$$\begin{bmatrix} E_1(x_3) \\ E_2(x_3) \end{bmatrix} = (C_a + \mathrm{i}k_a x_3 C_b) \begin{bmatrix} \tilde{E}_{1a} \\ \tilde{E}_{2a} \end{bmatrix} \exp\left(\mathrm{i}k_a x_3\right), \tag{8}$$

whose amplitude has a linear dependence on x_3.

We have verified that conditions A and B are not jointly satisfied in any dielectric medium with symmetric $\underline{\underline{\varepsilon}}$, if the real and the imaginary parts of $\underline{\underline{\varepsilon}}^{-1}$ share the same principal axes. Hence, Voigt wave propagation is not possible in any uniaxial medium, but it can occur in certain absorbing biaxial mediums [5, 6, 8]. It can also occur in magnetoplasmas [8], their permittivity dyadics comprising uniaxial as well as gyrotropic parts.

3. Conclusion

Condition A is the same as

$$\begin{aligned} &\varepsilon_{13}^2\varepsilon_{31}^2 + \varepsilon_{23}^2\varepsilon_{32}^2 - 2\varepsilon_{23}[2\varepsilon_{12}\varepsilon_{31} - (\varepsilon_{11} - \varepsilon_{22})\,\varepsilon_{32}]\varepsilon_{33} \\ &+ [(\varepsilon_{11} - \varepsilon_{22})^2 + 4\varepsilon_{12}\varepsilon_{21}]\,\varepsilon_{33}^2 \\ &+ 2\varepsilon_{13}\{\varepsilon_{23}\varepsilon_{31}\varepsilon_{32} - [2\varepsilon_{21}\varepsilon_{32} + (\varepsilon_{11} - \varepsilon_{22})\,\varepsilon_{31}]\,\varepsilon_{33}\} = 0\,, \end{aligned} \tag{9}$$

while condition B implies

$$\varepsilon_{12}\varepsilon_{33} \neq \varepsilon_{13}\varepsilon_{32} \quad \text{and/or} \quad \varepsilon_{21}\varepsilon_{33} \neq \varepsilon_{23}\varepsilon_{31}\,. \tag{10}$$

These two conditions are the chief contribution of this Communication, and do not appear to have been published before. Using these conditions, materials designers can explore and exploit the occurrence of Voigt wave propagation in novel dielectric materials. For that purpose, it would be convenient to express

$$\underline{\underline{\varepsilon}} = \underline{\underline{S}}(\theta, \phi) \cdot \underline{\underline{\varepsilon}}_{\mathrm{ref}} \cdot \underline{\underline{S}}^T(\theta, \phi) \tag{11}$$

and vary the angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, with $\underline{\underline{\varepsilon}}_{\mathrm{ref}}$ as the permittivity dyadic in some reference coordinate system and $\underline{\underline{S}}(\theta, \phi)$ as the 3-D rotation dyadic.

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