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General schema for the Brewster conditions

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General schema for the Brewster conditions. A matrix-based treatment of planewave reflection at *general* bimaterial interfaces has been used to obtain the Brewster conditions for the planar interface of two Lorentz-covariant media. There are two such conditions obtained, one for each half-space containing the incident planewave; this is remarkable since both Brewster conditions have turned out to be identical for all of the *specific* bimaterial interfaces examined in the past.

Allgemeine Darstellung der Brewster-Bedingungen. Eine Behandlung der Reflexion von ebenen Wellen an Übergängen zweier Materialien wird für den Erhalt der Brewsterbedingungen mittels Matrizen für planare Schnittstellen Lorentz-invarianter Medien durchgeführt. Es wurden zwei solcher Bedingungen erhalten, je eine für jeden eine einfallenden Planwelle enthaltender Halbraum; dies ist insbesondere bemerkenswert, da beide Brewster-Bedingungen sich als identisch für alle der in der Vergangenheit untersuchten spezifischer Übergänge zweier Materialien erwiesen haben.

Introduction

Around the end of the year 1812, Sir David Brewster had been successful in a series of experiments that elucidated the nature of polarized light. In particular, he had deduced the incidence condition for polarization by reflection: when unpolarized light is incident on an optically smooth, planar, air-dielectric interface at a certain angle with respect to the normal to the interface, the reflected light is completely polarized [1]. This particular angle came to be known as the *Brewster angle,* and the phenomenon has been widely utilized in the construction of Brewster polarizers [2].

It appears that during the 1950s, the definition of the Brewster angle changed from being a polarizing angle to that of a zero-reflection angle, at least operationally [3], the correct definition being found only in a few modern textbooks [e.g., 4]. Yet the correct definition is very powerful; indeed, it is pregnant with meaning [5] when compared with the sterility of the upstart zero-reflection definition.

Impelled by this driving force, the author has in the recent past investigated a *Brewster condition* for several planar bimaterial interfaces [5–11]. The investigations detailed in [5–11], however, suffer from one common defect: they are specific to the electromagnetic natures of the continua (material or vacuous) occupying the two halfspaces separated by the planar interface. It is quite clear, therefore, that though these studies have been helpful in stimulating thought, the conclusions drawn therefrom cannot be considered general enough, bearing the numerous varieties of electromagnetic continua possible [12] in mind.

The requirement of sufficient generality imposes a strain that can be handled only with a sufficiently powerful analytical procedure: this is provided by the Jones calculus so commonly used in optical system design [13, 14], but which has begun to be investigated thoroughly only recently [15–21] for more general material systems. In this communication matrix calculus is utilized to set up the electromagnetic fields on either side of a planar interface, the two halfspaces being assumed to be occupied by homogeneous, non-diffusive, linear bianisotropic media that are required for Lorentz-covariant representations of fields in continua [12]. In this fashion is obtained a general schema for the Brewster conditions.

Homogeneous Lorentz-covariant continua

By requiring that all field equations be generally covariant, Post [12] obtained constitutive equations for spatially local, linear, non-diffusive, homogeneous materials. In the frequency domain [$\exp(-i\omega t)$], these relations can be specified as

$$\boldsymbol{D} = \mathfrak{e}_P \cdot \boldsymbol{E} + \mathfrak{a}_P \cdot \boldsymbol{B}, \qquad \boldsymbol{H} = \mathfrak{b}_P \cdot \boldsymbol{E} + \mathfrak{m}_P^{-1} \cdot \boldsymbol{B}, \quad (1\,\text{a, b})$$

in which $\mathfrak{e}_P$, etc., are three-dimensional cartesian tensors of the second rank or dyadics, while $\mathfrak{m}_P^{-1}$ is the inverse of $\mathfrak{m}_P$. Relations (1 a, b) are not very convenient for the satisfaction of boundary value problems, which require the continuities of the tangential $\boldsymbol{E}$ and the tangential $\boldsymbol{H}$ fields; therefore, by making the transformation

$$\{\mathfrak{e} = \mathfrak{e}_P - \mathfrak{a}_P \cdot \mathfrak{m}_P \cdot \mathfrak{b}_P,\ \mathfrak{a} = \mathfrak{a}_P \cdot \mathfrak{m}_P,$$
$$\mathfrak{b} = -\,\mathfrak{m}_P \cdot \mathfrak{b}_P,\ \mathfrak{m} = \mathfrak{m}_P\}\,, \quad (2)$$

we obtain the equivalent Tellegen representation [22]

$$\boldsymbol{D} = \mathfrak{e} \cdot \boldsymbol{E} + \mathfrak{a} \cdot \boldsymbol{H}, \qquad \boldsymbol{B} = \mathfrak{b} \cdot \boldsymbol{E} + \mathfrak{m} \cdot \boldsymbol{H}, \quad (3\,\text{a, b})$$

that is more appropriate for the present purposes.

We begin the source-free Maxwell curl postulates

$$\nabla \times \boldsymbol{E} = i\omega \boldsymbol{B}, \qquad -\nabla \times \boldsymbol{H} = i\omega \boldsymbol{D}, \quad (4\,\text{a, b})$$

and without any significant loss of generality, substitute the spatial Fourier decomposition [23]

$$\boldsymbol{E}(x, y, z) = \boldsymbol{e}(z) \exp(i\kappa x), \quad (5\,\text{a})$$

$$\boldsymbol{H}(x, y, z) = \boldsymbol{h}(z) \exp(i\kappa x), \quad (5\,\text{b})$$

along with (3 a, b) in (4 a, b). Here, κ is a real number to be used later in order to ensure the satisfaction of Snel's

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law across the $z = 0$ interface, while

$$e(z) = e_x(z)\, u_x + e_y(z)\, u_y + e_z(z)\, u_z\,, \tag{6a}$$

$$h(z) = h_x(z)\, u_x + h_y(z)\, u_y + h_z(z)\, u_z\,, \tag{6b}$$

with u_x, u_y and u_z being the cartesian unit vectors.

These manipulations result in two algebraic equations, viz.,

$$e_y(z) = (\omega/\kappa)\, u_z \cdot [\mathfrak{b} \cdot e(z) + \mathfrak{m} \cdot h(z)]\,, \tag{7a}$$

$$h_y(z) = -(\omega/\kappa)\, u_z \cdot [\mathfrak{e} \cdot e(z) + \mathfrak{a} \cdot h(z)]\,, \tag{7b}$$

and the four differential equations,

$$\{d/dz\}\, e_x(z) = i\kappa e_z(z) + i\omega u_y \cdot [\mathfrak{b} \cdot e(z) + \mathfrak{m} \cdot h(z)]\,, \tag{8a}$$

$$\{d/dz\}\, e_y(z) = -i\omega u_x \cdot [\mathfrak{b} \cdot e(z) + \mathfrak{m} \cdot h(z)]\,, \tag{8b}$$

$$\{d/dz\}\, h_x(z) = i\kappa h_z(z) - i\omega u_y \cdot [\mathfrak{e} \cdot e(z) + \mathfrak{a} \cdot h(z)]\,, \tag{8c}$$

$$\{d/dz\}\, h_y(z) = i\omega u_x \cdot [\mathfrak{e} \cdot e(z) + \mathfrak{a} \cdot h(z)]\,, \tag{8d}$$

Assuming that (7a) and (7b) are linearly independent when $e_z(z)$ and $h_z(z)$ are the two unknowns, and substituting their consequent solution into (8a–d), we obtain the matrix differential equation

$$\{d/dz\}\, [f(z)] = i\, [P]\, [f(z)]\,, \tag{9}$$

in which

$$[f(z)] = \text{column}\, [e_x(z); e_y(z); h_x(z); h_y(z)] \tag{10}$$

is a column 4-vector, while $[P]$ is a 4×4 matrix that depends on $\mathfrak{e}$, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{m}$ and κ.

It is the eigenvectors of $[P]$ that interest us. Since $[P]$ is a 4×4 matrix, we invoke 4 linearly independent eigenvectors $[\gamma_m]$, $(m = 1, 2, 3, 4)$ such that

$$[P]\, [\gamma_m] = \lambda_m\, [\gamma_m]\,, \quad m = 1, 2, 3, 4, \tag{11}$$

where the λ_m are the corresponding eigenvalues of $[P]$; the invocation can be done from purely physical arguments. It is not necessary that $[P]$ have all of its eigenvalues distinct; instead, it is necessary and sufficient [24, 25] to have four linearly independent eigenvectors in order to diagonalize $[P]$.

The solution of (9) can be obtained as [20]

$$[f(z)] = [G]\, [L(z)]\, [G]^{-1}\, [f(0)]\,, \tag{12}$$

where the columns of the 4×4 matrix $[G]$ are the eigenvectors $[\gamma_m]$ arranged as per

$$[G] = [[\gamma_1]; [\gamma_2]; [\gamma_3]; [\gamma_4]]\,, \tag{13}$$

while $[L(z)]$ is the diagonal matrix

$$[L(z)] = \text{diagonal}\, [\exp(i\lambda_1 z); \exp(i\lambda_2 z); \exp(i\lambda_3 z); \exp(i\lambda_4 z)]\,. \tag{14}$$

Since this medium must allow wave propagation in all directions, and because the constitutive tensors can be set down in biaxial forms, we can order the eigenvalues such that

$$\text{Real}\{\lambda_1\} > 0\,, \quad \text{Real}\{\lambda_2\} > 0\,,$$
$$\text{Real}\{\lambda_3\} < 0\,, \quad \text{Real}\{\lambda_4\} < 0\,, \tag{15}$$

for what follows.

The solution (12) is uniquely determined by the boundary value $[f(0)]$, while $[G]^{-1}\, [f(z)]$ form the field eigenstates; thus, we can have

$$[f(0)] = [G]\, [c]\,, \qquad [f(z)] = [G]\, [L(z)]\, [c]\,, \tag{16a, b}$$

where $[c]$ is a 4-vector consisting of coefficients of expansion. It is advantageous to partition $[G]$ and $[c]$ as

$$[G] = \begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix}, \qquad [c] = \begin{bmatrix} [c^{(+)}] \\ [c^{(-)}] \end{bmatrix}, \tag{17a, b}$$

where $[A]$, $[B]$, $[C]$ and $[D]$ are 2×2 matrices while $[c^{(\pm)}] = \text{column}\, [c_1^{(\pm)}; c_2^{(\pm)}]$ are 2-vectors. In view of (15) and (16b), the 2-vector $[c^{(+)}]$ represents planewaves traveling towards $z = \infty$, while the 2-vector $[c^{(-)}]$ is for planewaves traveling towards $z = -\infty$.

Reflection matrices

Let now the plane $z = 0$ separate the two media characterized by

$$D = \mathfrak{e}_q \cdot E + \mathfrak{a}_q \cdot H\,, \quad z \le 0 \tag{18a}$$

$$B = \mathfrak{b}_q \cdot E + \mathfrak{m}_q \cdot H\,, \quad z \le 0 \tag{18b}$$

$$D = \mathfrak{e}_s \cdot E + \mathfrak{a}_s \cdot H\,, \quad z \ge 0 \tag{18c}$$

$$B = \mathfrak{b}_s \cdot E + \mathfrak{m}_s \cdot H\,, \quad z \ge 0 \tag{18d}$$

and we have that the representations (6a, b) are adequate for the fields in the two halfspaces; thus κ is the common horizontal wave number mandated by Snel's law or the phase-matching condition. Consequently, the field behavior is adequately described by (16a, b) in both halfspaces, and we have

$$[f(z)] = [G_q]\, [L_q(z)]\, [q]\,, \quad z \le 0\,, \tag{19a}$$

$$[f(z)] = [G_s]\, [L_s(z)]\, [s]\,, \quad z \ge 0\,, \tag{19b}$$

where $[q]$ and $[s]$ are 4-vectors. The continuities of the tangential components of the E and the H fields across the interface $z = 0$ require that

$$[f(0+)] = [f(0-)]\,. \tag{20}$$

We begin with planewave incidence from the halfspace $z \le 0$; in other words, we set $[s^{(-)}] = \text{column}\, [0; 0]$; then, from (19a, b) and using partitions of the type (17a, b) we obtain from (20) that

$$[q^{(-)}] = [R_{qq}]\, [q^{(+)}]\,, \tag{21a}$$

where the 2×2 reflection matrix $[R_{qq}]$ can be computed from

$$[R_{qq}] = -[D_q]^{-1}\, [\Delta]^{-1}\, [A_q]\, [\Delta_q]\,. \tag{21b}$$

In a similar fashion, when we have $[q^{(+)}] = \text{column}\, [0; 0]$ and the incidence is from the halfspace $z \ge 0$, we get

$$[s^{(+)}] = [R_{ss}]\, [s^{(-)}]\,, \tag{22a}$$

in which the 2×2 reflection matrix $[R_{ss}]$ is given as

$$[R_{ss}] = -[C_s]^{-1}\, [\Delta]^{-1}\, [B_s]\, [\Delta_s]\,. \tag{22b}$$

Here,

$$[\Delta] = [B_q][D_q]^{-1} - [A_s][C_s]^{-1}, \tag{23a}$$

$$[\Delta_q] = [I] - [A_q]^{-1}[A_s][C_s]^{-1}[C_q], \tag{23b}$$

$$[\Delta_s] = [I] - [B_s]^{-1}[B_q][D_q]^{-1}[D_s], \tag{23c}$$

with $[I]$ being the identity matrix.

The Brewster conditions

We note from (21 a) that the reflection ratio $[q_1^{(-)}/q_2^{(-)}]$ is independent of the incidence ratio $[q_1^{(+)}/q_2^{(+)}]$ provided the matrix $[R_{qq}]$ is singular, i.e., the determinant of $[R_{qq}]$ equals zero. Therefore, for incidence from the $z \le 0$ half space, if κ_q is such that

$$\{(\det[A_q]\cdot\det[\Delta_q])/(\det[D_q]\cdot\det[\Delta])\}\,|_{\kappa=\kappa_q} = 0, \tag{24}$$

then κ_q is the Brewster wavenumber and (24) is the corresponding Brewster condition; here det denotes the determinant.

Likewise from (22a), the reflection ratio $[s_1^{(+)}/s_2^{(+)}]$ is independent of the incidence ratio $[s_1^{(-)}/s_2^{(-)}]$ if the matrix $[R_{ss}]$ is singular. In other words, for incidence from the halfspace $z \ge 0$, if κ_s is such that

$$\{(\det[B_s]\cdot\det[\Delta_s])/(\det[C_s]\cdot\det[\Delta])\}\,|_{\kappa=\kappa_s} = 0, \tag{25}$$

then κ_s is the Brewster wavenumber and (25) is the corresponding Brewster condition.

Thus, depending on which halfspace the incidence is from, we have a Brewster condition that tells us when the ratio of the reflection amplitudes is totally decorrelated from the ratio of the incidence amplitudes. It is to be noted that (24) and (25) do not explicitly depend on the phase velocities in the two media involved.

In all cases examined so far [5–10], it has been that (24) and (25) turn out be identical conditions and $\kappa_s = \kappa_q$; these cases have included isotropic media, uniaxial dielectric media and nonreciprocal media. However, in the general schema presented here, no further simplification of (24) and (25) may be possible without a more informative specification of the constitutive dyadics.

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