Negative–phase–velocity propagation of electromagnetic waves in charged rotating black holes

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Abstract. The propensity of a rotating black hole to support the propagation of electromagnetic plane waves with negative phase velocity (NPV) in its ergosphere is highly sensitive to the presence of charge, whether the Kerr–Newman or the Kerr–Sen metric descriptions of spacetime are considered. The change in shape and size of the ergosphere induced by increasing the black hole’s charge give rise to an increase the possibilities for NPV propagation. Increasing the black hole’s charge at a fixed surface concentrates the NPV–supporting regions towards the ergosphere’s equator. Striking differences in the NPV characteristics for the Kerr–Newman and Kerr–Sen metrics emerge from numerical studies, particularly close to the outer event horizon when the magnitude of the charge is large.

Keywords. Negative phase velocity (NPV), ergosphere, Kerr–Newman, Kerr–Sen

PACS Nos 04.20.-q, 42.25.Bs, 98.80.-k

1. Introduction

Electromagnetic plane waves with negative phase velocity (NPV) have been the center of much attention lately [1]. They give rise to many exotic phenomena — the most prominent being negative refraction [2] — which are not associated with their positive–phase–velocity (PPV) counterparts. Much of the interest in NPV propagation has stemmed from the realization of negatively refracting metamaterials [3]. In addition, the prospects of NPV propagation in special [4] and general [5] relativistic scenarios, and the concomitant implications for observational astronomy, have prompted study and speculation within the past few years [6].

It is now theoretically well–established that, for certain spacetime metrics, NPV propagation can be supported in vacuum [5,7]. In particular, NPV can arise in the ergosphere of a rotating black hole described by the Kerr metric [8]. The NPV–supporting regions are concentrated towards the equator of the ergosphere, and this concentration becomes exaggerated as the angular velocity of the black hole is increased.

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increased. A plausible astrophysical scenario in which NPV propagation may occur has been put forward by Punsly & Coroniti [9]. They describe interactions between a large-scale magnetic field and a plasma inside the ergosphere of a rotating black hole which results in the creation of photons that appear to certain observers to have negative energy and be “rotating backward”. Furthermore, the production of these negative-energy photons is concentrated in the equatorial region [10].

Black holes are characterized by three quantities — mass, angular momentum, and charge — the third of which has not been considered so far in the context of NPV propagation. Here we address the question: how does the presence of charge effect the propensity of a rotating black hole to support NPV propagation?

It is debatable whether rotating black holes in astrophysical settings are charged, as has been discussed at length [11]. While we do not enter this debate here, we do note that charged rotating black holes plausibly arise following the catastrophic gravitational collapse of the most massive magnetized rotating stars [11]. The existence of charged rotating black holes is not forbidden on experimental or theoretical grounds [12]. Indeed, charged black holes continue to be a topic of intense research interest, both experimentally [13] and theoretically [14–16]. For example, the observed features of gamma-ray bursts can be modeled by the presence of an electrically charged back hole [17]. Other recent studies consider charged black holes in connection with collapsing stars [18,19], spacetime curvature [20] and cosmic dust [21,22].

The most commonly encountered spacetime description of a charged rotating black hole is provided by the Kerr–Newman metric [23]. This solution to the Einstein field equations was derived in the 1960s [24,25]. In 1992, Sen developed an alternative metric solution describing a charged rotating black hole [26]. The approach was based on the action of a heterotic string in the low-energy limit, applied to the classical Kerr solution. Lately, the singularity [27], thermodynamic [28], geodesic [29] and quantum thermal [30] properties of this Kerr–Sen metric have been established. In the following sections, we consider both the Kerr–Newman and the Kerr–Sen metrics in our investigation of the NPV characteristics of a charged rotating black hole.

2. Theoretical framework

2.1 Preliminaries

Electromagnetic propagation is considered in free space. The spacetime curvature is described by the metric $g_{\alpha\beta}$ with signature $(+,-,-,-)$. Here we outline the theory of gravitationally assisted NPV propagation, comprehensive details being available elsewhere [7,8].

Greek indexes take the values 0, 1, 2 and 3, Roman indexes take the values 1, 2 and 3, $x^0 = ct$, where $c$ is the speed of light in vacuum in the absence of all gravitational fields, and $x^{1,2,3}$ represent the three spatial coordinates.
Following the approach originally put forth by Tamm [31,32], we exploit the fact that electromagnetic propagation in curved spacetime in free space may be represented as propagation in a fictitious, instantaneously responding, bianisotropic medium in flat spacetime, characterized by the constitutive relations

\[
D_\ell = \gamma_{\ell m} E_m + \varepsilon_{\ell mn} \Gamma_m H_n, \quad B_\ell = \gamma_{\ell m} H_m - \varepsilon_{\ell mn} \Gamma_m E_n.
\]  

(1)

Herein, \(E_\ell, B_\ell, D_\ell, \) and \(H_\ell\) are the components of the conventional electromagnetic field vectors; \(\varepsilon_{\ell mn}\) is the three-dimensional Levi-Civita symbol; the tensor \(\gamma_{\ell m} = -\sqrt{-g} (g_{\ell m}/g_{00})\) and the vector \(\Gamma_m = g_{0m}/g_{00}\).

In order to consider planewave propagation, we rewrite (1) in \(3 \times 3\) dyadic/3 vector form as

\[
\begin{align*}
D(\text{ct}, \mathbf{r}) &= \epsilon_0 \gamma(\text{ct}, \mathbf{r}) \cdot E(\text{ct}, \mathbf{r}) - c^{-1} \Gamma(\text{ct}, \mathbf{r}) \times H(\text{ct}, \mathbf{r}) \\
B(\text{ct}, \mathbf{r}) &= \mu_0 \gamma(\text{ct}, \mathbf{r}) \cdot H(\text{ct}, \mathbf{r}) + c^{-1} \Gamma(\text{ct}, \mathbf{r}) \times E(\text{ct}, \mathbf{r})
\end{align*}
\]

(2)

wherein \(\gamma(\text{ct}, \mathbf{r})\) is the dyadic–equivalent of \(\gamma_{\ell m}\), \(\Gamma(\text{ct}, \mathbf{r})\) is the vector–equivalent of \(\Gamma_m\), the scalar constants \(\epsilon_0\) and \(\mu_0\) denote the permittivity and permeability of vacuum in the absence of a gravitational field, and SI units are adopted. Let us emphasize that, while the spatial \(r\) and temporal \(t\) coordinates have been separated in (2), the underlying spacetime remains curved.

Our aim is to explore electromagnetic planewave propagation in the spacetime of a black hole of geometric mass \(m_{bh}\) and charge \(q_{bh}\), rotating with angular momentum \(a_{bh}\) about the Cartesian \(z\) axis, in conventional units. Two different descriptions of this spacetime are considered, as provided by the Kerr–Newman metric and the Kerr–Sen metric. The line elements for these are conventionally expressed in terms of Boyer–Lindquist coordinates. In order to establish the corresponding forms of the dyadic \(\gamma\) and vector \(\Gamma\), the Kerr–Schild Cartesian coordinate representations for these metrics are presented in the following two subsections.

2.2 Kerr–Newman spacetime

The Kerr–Newman line element is given in Boyer–Lindquist coordinates as [33,34]

\[
ds^2 = \Delta_{\text{KN}} \left( R^2 + a_{bh}^2 \cos^2 \theta_{\text{KN}} \right)^{-1} \left( dt - a_{bh} \sin^2 \theta_{\text{KN}} \, d\phi_{\text{KN}} \right)^2 \\
- \sin^2 \theta_{\text{KN}} \left( R^2 + a_{bh}^2 \cos^2 \theta_{\text{KN}} \right)^{-1} \left[ \left( R^2 + a_{bh}^2 \right) d\phi_{\text{KN}} - a_{bh} \, dt \right]^2 \\
- \left( R^2 + a_{bh}^2 \cos^2 \theta_{\text{KN}} \right) \Delta_{\text{KN}}^{-1} dR^2 - \left( R^2 + a_{bh}^2 \cos^2 \theta_{\text{KN}} \right) d\theta_{\text{KN}}^2.
\]

(3)

where \(R\) is given implicitly via \(R^2 = x^2 + y^2 + z^2 - a_{bh}^2 \left[ 1 - (z/R)^2 \right]\) and \(\Delta_{\text{KN}} = R^2 - 2m_{bh} R + a_{bh}^2 + q_{bh}^2\). By applying the coordinate transformation \((t, \phi_{\text{KN}}, R, \theta_{\text{KN}}) \rightarrow (t, x, y, z)\) specified by

\[
\begin{align*}
x &= R \cos \phi_{\text{KN}} + a_{bh} \sin \phi_{\text{KN}} \sin \theta_{\text{KN}}, \\
y &= R \sin \phi_{\text{KN}} - a_{bh} \cos \phi_{\text{KN}} \sin \theta_{\text{KN}}, \\
z &= R \cos \theta_{\text{KN}}
\end{align*}
\]

(4)
with \( \text{d}t = dt + dR - \left[ (R^2 + a_{\text{bh}}^2) / \Delta_{\text{KN}} \right] dR \) and \( \text{d}\phi_{\text{KN}} = d\phi_{\text{KN}} - (a_{\text{bh}} / \Delta_{\text{KN}}) dR \), the line element (3) is expressed in Kerr–Schild Cartesian form as

\[
\text{d}s^2 = \text{d}t^2 - d\tau^2 - dy^2 - d\zeta^2 - (2m_{\text{bh}} R - q_{\text{bh}}^2) R^2 \left( R^4 + a_{\text{bh}}^2 z^2 \right)^{-1} \times \left\{ \text{d}t - \left( z / R \right) d\zeta - \left( R^2 + a_{\text{bh}}^2 \right)^{-1} \left[ R (x dx + y dy) - a_{\text{bh}} (x dy - y dx) \right] \right\}^2 .
\]  

The corresponding metric components \( g_{\alpha\beta} \) — from which the associated dyadic \( \gamma \) and vector \( \Gamma \) immediately follow — are provided elsewhere [35].

We focus on the regime \( m_{\text{bh}}^2 > a_{\text{bh}}^2 + q_{\text{bh}}^2 \). Furthermore, our interest lies in the ergosphere demarcated as \( R_+ < R < R_{S+} \), with \( R = R_+ \) representing the outer event horizon and \( R = R_{S+} \) representing the stationary limit surface [23]. The outer event horizon occurs where \( \Delta_{\text{KN}} = 0 \). Thus, \( R_+ = m_{\text{bh}} + \sqrt{m_{\text{bh}}^2 - a_{\text{bh}}^2 - q_{\text{bh}}^2} \).

The stationary limit surface \( R = R_{S+} \) occurs where \( g_{00} = 0 \). Thus, \( R_{S+} = m_{\text{bh}} + \sqrt{m_{\text{bh}}^2 - a_{\text{bh}}^2 (z/R)^2} \).

### 2.3 Kerr–Sen spacetime

We now turn to the Kerr–Sen line element which is given in Boyer–Lindquist coordinates as [28,30]

\[
\text{d}s^2 = \Delta_{\text{KS}} \left( \tilde{S}^2 + a_{\text{bh}}^2 \cos^2 \theta_{\text{KS}} \right)^{-1} \left( dt - a_{\text{bh}} \sin^2 \theta_{\text{KS}} d\phi_{\text{KS}} \right)^2
- \sin^2 \theta_{\text{KS}} \left( \tilde{S}^2 + a_{\text{bh}}^2 \cos^2 \theta_{\text{KS}} \right)^{-1} \left[ (\tilde{S}^2 + a_{\text{bh}}^2) d\phi_{\text{KS}} - a_{\text{bh}} dt \right]^2
- \left( \tilde{S}^2 + a_{\text{bh}}^2 \cos^2 \theta_{\text{KS}} \right) \Delta_{\text{KS}}^{-1} d\tilde{S}^2 - \left( \tilde{S}^2 + a_{\text{bh}}^2 \cos^2 \theta_{\text{KS}} \right) d\tilde{\theta}_{\text{KS}}^2
\]

where \( \Delta_{\text{KS}} = S^2 - 2 \left( m_{\text{bh}} - (\beta / 2) \right) S + a_{\text{bh}}^2, \tilde{S}^2 = S^2 + \beta S, \beta = q_{\text{bh}}^2 / m_{\text{bh}}, and \tilde{S} \) satisfies \( \tilde{S}^2 = x^2 + y^2 + z^2 - a_{\text{bh}}^2 \left[ 1 - (z/\tilde{S})^2 \right] \). Under the coordinate transformation \((t, \phi_{\text{KS}}, S, \theta_{\text{KS}} \longrightarrow (\tilde{t}, x, y, z)\) specified by

\[
x = (\tilde{S} \cos \phi_{\text{KS}} + a_{\text{bh}} \sin \phi_{\text{KS}}) \sin \theta_{\text{KS}},
y = (\tilde{S} \sin \phi_{\text{KS}} - a_{\text{bh}} \cos \phi_{\text{KS}}) \sin \theta_{\text{KS}},
z = \tilde{S} \cos \theta_{\text{KS}}
\]

with \( \text{d}\tilde{t} = dt + d\tilde{S} - \left[ (\tilde{S}^2 + a_{\text{bh}}^2) / \Delta_{\text{KS}} \right] d\tilde{S} \) and \( \text{d}\phi_{\text{KS}} = d\phi_{\text{KS}} - (a_{\text{bh}} / \Delta_{\text{KS}}) d\tilde{S} \), the Kerr–Schild Cartesian form of the line element (6) emerges as

\[
\text{d}s^2 = \text{d}\tilde{t}^2 - d\tau^2 - dy^2 - d\zeta^2 + \frac{\beta^2 \left( \tilde{S}^2 + a_{\text{bh}}^2 \cos^2 \theta_{\text{KS}} \right)}{(\beta^2 + 4 \tilde{S}^2) \Delta_{\text{KS}}} d\tilde{S}^2 - \frac{2m_{\text{bh}} \tilde{S} S}{S^4 + a_{\text{bh}}^2 \tilde{S}^2} \times \left\{ \text{d}\tilde{t} - \left( z / \tilde{S} \right) d\zeta - \left( S^2 + a_{\text{bh}}^2 \right)^{-1} \left[ \tilde{S} (x dx + y dy) - a_{\text{bh}} (x dy - y dx) \right] \right\}^2 .
\]

Explicit expressions for the corresponding metric components \( g_{\alpha\beta} \) are presented elsewhere [35]. From these, the associated \( \gamma \) and \( \Gamma \) immediately follow.

In an analogous manner to the case for the Kerr–Newman metric, the outer event horizon arises at \( S = S_+ = m_{\text{bh}} - (\beta / 2) + \sqrt{[m_{\text{bh}} - (\beta / 2)]^2 - a_{\text{bh}}^2} \).
whereas the stationary limit surface arises at $S = S_{S_+} = m_{bh} - (\beta/2) + \sqrt{[m_{bh} - (\beta/2)]^2 - (a_{bh}z/S_{S_+})^2}$. As in §2.2, the ergosphere is the region bounded by the outer event horizon and the stationary limit surface; i.e., $S_+ < S < S_{S_+}$.

### 2.4 Planewave propagation

Now we turn to the propagation of plane waves in the medium characterized by (2). By choosing a sufficiently small neighbourhood $\mathcal{R}$ containing the point $\tilde{r} = \tilde{x} \hat{x} + \tilde{y} \hat{y} + \tilde{z} \hat{z}$ in the Kerr–Schild Cartesian coordinate system, we approximate the nonuniform metric $g_{\alpha\beta}$ with the uniform metric $\tilde{g}_{\alpha\beta}$ throughout $\mathcal{R}$ [5]. Within the neighbourhood $\mathcal{R}$, we utilize the uniform $3\times3$ dyadic $\tilde{\gamma} \equiv \left. \gamma \right|_{\mathcal{R}}$ and the uniform $3$ vector $\tilde{\Gamma} \equiv \left. \Gamma \right|_{\mathcal{R}}$. Due to the uniform approximation, attention is restricted to electromagnetic wavelengths that are small compared to the linear dimensions of $\mathcal{R}$. Whereas the coordinate system remains global (and spacetime therefore remains curved), in the ensuing developments the planewave representation is not global but is neighbourhood–specific instead. The neighbourhood–specific reference frame should be distinguished from a local Minkowskian reference frame. Any transformation from the neighbourhood–specific frame to a local frame eliminates the influence of gravity and may induce a confusion between space and time [36,37].

The three–dimensional Fourier transforms

$$\begin{align*}
\mathcal{E}(ct, \mathbf{r}) &= \frac{1}{c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}(\omega/c, \mathbf{k}) \exp \{i(\mathbf{k} \cdot \mathbf{r} - \omega t)\} \, d\omega \, dk_1 \, dk_2, \\
\mathcal{H}(ct, \mathbf{r}) &= \frac{1}{c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{H}(\omega/c, \mathbf{k}) \exp \{i(\mathbf{k} \cdot \mathbf{r} - \omega t)\} \, d\omega \, dk_1 \, dk_2,
\end{align*}$$

(9)

decompose the electromagnetic fields within $\mathcal{R}$. Here, the wavevector $\mathbf{k}$ is the Fourier variable corresponding to $\mathbf{r}$, $i = \sqrt{-1}$ and $\omega$ is the usual temporal frequency.

By combining the Maxwell curl postulates and (9), and then solving the resulting $4 \times 4$ eigenvalue problem, the wavevector component $k_3$ is determined [38]. Only propagating (non–evanescent) plane waves are considered; i.e., $k_3 \in \mathbb{R}$.

NPV propagation occurs when the phase velocity vector casts a negative projection on to the time-averaged Poynting vector — i.e.,

$$\mathbf{k} \cdot \langle \mathcal{P}(\omega/c, \mathbf{k}) \rangle_t < 0,$$

(10)

where the time–averaged Poynting vector is calculated from the complex–valued phasors of the electric and magnetic fields, $\mathcal{E}(\omega/c, \mathbf{k})$ and $\mathcal{H}(\omega/c, \mathbf{k})$, as per

$$\langle \mathcal{P}(\omega/c, \mathbf{k}) \rangle_t = \frac{1}{2} \Re \{ \mathcal{E}(\omega/c, \mathbf{k}) \times \mathcal{H}^*(\omega/c, \mathbf{k}) \},$$

(11)

$^{2}\Re \{ \cdot \}$ denotes the real part while the asterisk denotes the complex conjugate.
as described in detail elsewhere [5,7,8], a standard plane-wave analysis provides the sufficient condition for NPV propagation

\[ P_a = \left( \mathbf{\hat{e}}_a \cdot \mathbf{\hat{\gamma}} \times \mathbf{\hat{e}}_a \right) \left( k \cdot \mathbf{\hat{\gamma}} \cdot \mathbf{p} \right) < 0, \quad P_b = \left( \mathbf{\hat{e}}_b \cdot \mathbf{\hat{\gamma}} \times \mathbf{\hat{e}}_b \right) \left( k \cdot \mathbf{\hat{\gamma}} \cdot \mathbf{p} \right) < 0, \quad (12) \]

wherein the unit vectors

\[ \mathbf{e}_a = \mathbf{\hat{\gamma}}^{-1} \cdot \mathbf{w} \left( \left( \mathbf{\hat{\gamma}}^{-1} \cdot \mathbf{w} \right)^{-1} \right), \quad \mathbf{e}_b = \mathbf{\hat{\gamma}}^{-1} \cdot \left( \mathbf{p} \times \mathbf{e}_a \right) \left( \left( \mathbf{\hat{\gamma}}^{-1} \cdot \left( \mathbf{p} \times \mathbf{e}_a \right) \right)^{-1} \right) \quad (13) \]

and \( \mathbf{p} = k - (\omega/c) \hat{L} \). The unit vector \( \mathbf{w} \) is required to be orthogonal to \( \mathbf{p} \), i.e., \( \mathbf{w} \cdot \mathbf{p} = 0 \), but is otherwise arbitrary. The two wavenumbers \( k = k^\pm \) given by

\[ k^\pm = \frac{\omega}{c} \left[ k \cdot \mathbf{\hat{\gamma}} \cdot \hat{L} \pm \sqrt{\left( k \cdot \mathbf{\hat{\gamma}} \cdot \hat{L} \right)^2 - k \cdot \mathbf{\hat{\gamma}} \cdot \hat{L} \left( \hat{L} \cdot \mathbf{\hat{\gamma}} \cdot \hat{L} - |\mathbf{\hat{\gamma}}| \right)} \right] \left( \mathbf{\hat{\gamma}} \cdot \mathbf{\hat{\gamma}} \right)^{-1}, \quad (14) \]

are associated with the arbitrarily oriented wavevector \( \mathbf{k} = \hat{k}k \) with \( \hat{k} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta \) in Kerr–Schild coordinates.

Finally, in this section, let us describe the wavelength regime in which the NPV condition (12) holds. For the ergosphere region of the Kerr–Newman black hole, 

\[ m_{bh} + \sqrt{m_{bh}^2 - a_{bh}^2 - q_{bh}^2} < R < m_{bh} + \sqrt{m_{bh}^2 - q_{bh}^2} \]  

so that \( (R/m_{bh}) > 1 > (q_{bh}/m_{bh}) \). Hence, \( \Delta_{KN} \approx R^2 - 2m_{bh}R + a_{bh}^2 \), the approximation being better at relatively higher values of the angular momentum. To leading order, the curvature of the Kerr–Newman spacetime can therefore be approximated by that of the Kerr spacetime. Consequently, the NPV condition (12) holds in the short–wavelength regime \( (2\pi/|k|) \ll R \) [7]. In a similar fashion, since \( \Delta_{KS} \approx S^2 - 2m_{bh}S + a_{bh}^2 \), the short–wavelength regime \( (2\pi/|k|) \ll S \) applies to the NPV condition (12) for the Kerr–Sen ergosphere. Thus, at wavelengths of the order of a kilometre or less, the short wavelength regimes apply for black holes of the mass of our sun.

3. Numerical Results

The NPV characteristics of an uncharged rotating black hole, as described by the Kerr metric, have been established previously [8]. In this section, we report on the effect that charge has on the propensity for a rotating black hole to support NPV propagation in its ergosphere. This study was carried out by numerically evaluating \( \mathbf{k} \cdot \left( \mathbf{P} / (\omega/c, \mathbf{k}) \right) \), for both the Kerr–Newman and Kerr–Sen descriptions of the black hole spacetime.

The following strategy was implemented: for fixed values of \( m_{bh}, a_{bh}, \) and \( q_{bh} \), we considered surfaces of constant \( R \) for the Kerr–Newman metric, and surfaces of constant \( S \) for the Kerr–Sen metric, with all surfaces being contained within the respective black hole ergospheres. The prospects for NPV propagation for all propagation directions \( \hat{k} = \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta \) were calculated over a grid of points on surfaces of constant \( R \) and \( S \). In each neighbourhood \( R \)
containing the point \( \hat{x} \hat{x} + \hat{y} \hat{y} + \hat{z} \hat{z} \) on the grid, the quantities \( P_a \) and \( P_b \) were computed at 1° increments for \( \theta_a \in [0°, 180°) \) and \( \phi_a \in [0°, 360°) \), for both \( k = k^+ \) and \( k = k^- \). For each value of \((\theta_a, \phi_a)\) at which NPV was found to be supported (i.e., \( P_a < 0 \) and \( P_b < 0 \)), a score of one point was assigned to the \((\hat{x}, \hat{y}, \hat{z})\) grid point. Thus, the total score represents the proportion of the \((\theta_a, \phi_a)\) parameter space which supports NPV propagation. The total score at each grid point was translated to a colour for the convenient presentation of results.

Fig. 1 shows the NPV propagation maps for Kerr–Newman spacetime for a charged rotating black hole with \( a_{bh} = m_{bh} \sqrt{3}/4 \). Three constant \( R \) surfaces within the ergosphere are considered, namely \( R = 0.99R_+ + 0.01R_{S_+} \) with \( q_{bh} = 0.1 \ m_{bh} \); (b) \( R = 0.5R_+ + 0.5R_{S_+} \), \( q_{bh} = 0.1 \ m_{bh} \); (d) \( R = 0.99R_+ + 0.01R_{S_+} \), \( q_{bh} = 0.45 \ m_{bh} \); (e) \( R = 0.5R_+ + 0.5R_{S_+} \), \( q_{bh} = 0.45 \ m_{bh} \); and (f) \( R = 0.01R_+ + 0.99R_{S_+} \), \( q_{bh} = 0.45 \ m_{bh} \). We note that \( q_{bh} \leq 0.5 \) in order for the ergosphere boundaries to remain real–valued.

In Fig. 1(a)–(c), for \( q_{bh} = 0.1 \ m_{bh} \), the regions of NPV propagation are most heavily concentrated near the equator of the ergosphere and NPV becomes less likely towards the poles. Also, NPV propagation is more likely near the outer event horizon and less likely near the stationary limit surface. Increasing the value of \( q_{bh} \) does not markedly change the qualitative nature of the NPV characteristics, as illustrated by Fig. 1(d)–(f). However, the incidence of NPV propagation is
considerably greater at \( q_{bh} = 0.45 \, m_{bh} \) as compared with \( q_{bh} = 0.1 \, m_{bh} \), at all three constant–\( R \) surfaces.

![Diagram showing NPV propagation maps for different values of \( q_{bh} \).](image)

**Figure 2.** As Fig. 1 but (a) at the surface \( R = 0.99 R^K_+ + 0.01 R^K_{S+} \) with \( q_{bh} = 0 \); (b) \( R = 0.99 R^K_+ + 0.01 R^K_{S+}, q_{bh} = 0.1 \, m_{bh} \); (c) \( R = 0.99 R^K_+ + 0.01 R^K_{S+}, q_{bh} = 0.45 \, m_{bh} \); (d) \( R = 0.5 R^K_+ + 0.5 R^K_{S+}, q_{bh} = 0 \); (e) \( R = 0.5 R^K_+ + 0.5 R^K_{S+}, q_{bh} = 0.1 \, m_{bh} \); and (f) \( R^K = 0.5 R^K_+ + 0.5 R^K_{S+}, q_{bh} = 0.45 \, m_{bh} \).

It is clear that increasing \( q_{bh} \) has the effect of reducing \( R_+ \) and \( R_{S+} \). Therefore, in connection with Fig. 1, the question arises: is the observable increase in NPV propensity as \( q_{bh} \) increases attributable to the corresponding decrease in \( R_+ \) and \( R_{S+} \)? To answer this question, we consider surfaces of constant \( R \) relative to the outer event horizon for the uncharged rotating black hole; i.e., \( R = R^K_+ = m_{bh} + \sqrt{m_{bh}^2 - a_{bh}^2} \), and the stationary limit surface for the uncharged rotating black hole; i.e., \( R = R^K_{S+} = \frac{m_{bh}}{3} + \sqrt{\left(\frac{m_{bh}}{3}\right)^2 - \frac{a_{bh}^2}{3}} \). In Fig. 2, the NPV propagation maps, for Kerr–Newman spacetime, are presented on the ergosphere surfaces \( R = 0.99 R^K_+ + 0.01 R^K_{S+} \) (in Figs 2(a), (b) and (c)) and \( R = 0.5 R^K_+ + 0.5 R^K_{S+} \) (in Figs 2(d), (e) and (f)). As in Fig. 1, \( a_{bh} = m_{bh} \sqrt{3/4} \). The values of \( q_{bh} \) are: 0 in Figs 2(a) and (d); 0.1 \( m_{bh} \) in Figs 2(b) and (e); and 0.45 \( m_{bh} \) in Figs 2(c) and (f). Thus, the maps in Figs 2(a) and (d) correspond to the Kerr black hole. Fig. 2 indicates that for a fixed value of \( R \), the overall propensity for NPV propagation decreases as \( q_{bh} \) increases and the NPV supporting regions become more concentrated in the equatorial region of the ergosphere.

We turn now to the Kerr–Sen metric. The NPV propagation maps corresponding to those in Fig. 1, but calculated using the Kerr–Sen metric, are presented in Fig. 3. A comparison of Fig. 3(a)–(c) with Fig. 1(a)–(c) reveals that the NPV characteristics of the Kerr–Sen and Kerr–Newman metrics are quite similar at \( q_{bh} = 0.1 \, m_{bh} \).
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However, stark differences between the Kerr–Sen and Kerr–Newman NPV characteristics emerge at \( q_{bh} = 0.45 \, m_{bh} \), particularly near to the outer event horizon. For the Kerr–Sen metric, we see in Fig. 3(d) that the distribution of NPV propagation is concentrated in a region between the ergosphere pole and equator, while a relatively low incidence occurs at the equator. This is quite different from the situation for the Kerr–Newman metric that is displayed in Fig. 1(d).

We note that the NPV propagation maps at \( q_{bh} = 0.1 \, m_{bh} \), for both the Kerr–Newman metric and Kerr–Sen metric, are quite similar to those reported for the uncharged rotating black hole [8]. This reflects the fact that both the Kerr–Newman metric and the Kerr–Sen metric become identical to the Kerr metric in the limit \( q_{bh} \to 0 \). Furthermore, regardless of the absence or presence of charge, NPV is not supported at the poles of the ergosphere.

4. Negative phase velocity and superradiance

In astrophysics, the term superradiance is associated with the spontaneous emission of positive–energy photons in the ergosphere of a rotating black hole [39,40]. Energy conservation requires an accompanying creation of negative–energy photons. We compare and contrast the negative–energy photon trajectories which arise in black–hole superradiance with the foregoing account of NPV propagation.
The common ground between NPV propagation and superradiance is that both are associated with negative energy densities in the ergosphere, as discerned by an observer at asymptotic infinity [41]. However, there are notable differences between these two phenomena. A crucial property of the negative-energy photons arising from superradiance is that their angular momentum is initially directed opposite to the angular momentum of the black hole. In contrast, the ergosphere supports NPV plane waves with angular momentum density directed parallel, as well as anti-parallel, to the black hole’s angular momentum.

By way of illustration, let us again consider the propagation of plane waves, as represented by (9), in a specific neighbourhood in the ergosphere of a black hole rotating with angular momentum $a_{bh} = m_{bh}\sqrt{3}/4$ about the Cartesian $z$ axis. For simplicity, the black hole is assumed to be uncharged. We focus on the angular momentum density of plane waves for a neighbourhood positioned in the equatorial plane, given by the Boyer–Lindquist coordinates $\phi_{KN} = \theta_{KN} = 0$.

To compute the planewave’s global angular momentum, we start with the time-domain electric and magnetic fields as per

$$E_p(ct, r) = \text{Re}[\mathbf{e}_p \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]]$$

$$H_p(ct, r) = \text{Re}[\mathbf{h}_p \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]]$$

(15)

The vector cross product of these provides the instantaneous Poynting vector

$$P_{inst}^p(ct, r) = [\cos(\mathbf{k} \cdot \mathbf{r} - \omega t)]^2 \mathbf{e}_p \times \mathbf{h}_p.$$  

(16)

The planewave’s angular momentum about the $\hat{z}$ axis is given as [42]

$$L_p = \hat{z} \cdot \left[ \frac{1}{2} \frac{\partial}{\partial \phi_{KN}} (\mathbf{e}_p \times \mathbf{h}_p) \right].$$  

(18)

In Fig. 4, the sign of $L_p$ is mapped against the wavevector orientation angles $\theta_k$ and $\phi_k$ for $k^-$ NPV plane waves. Three neighbourhoods — specified by $R = 1.51m_{bh}$, $R = 1.75m_{bh}$ and $R = 1.9m_{bh}$ — in the equatorial plane of the ergosphere are considered in Fig. 4. The outer event horizon lies at $R_{+}^{\text{K}} = 1.5m_{bh}$ and the stationary limit surface at $R_{\text{S}}^{\text{K}} = 2m_{bh}$. In Fig. 4, $\Xi = 4$ means $L_a > 0$ and $L_b > 0$; $\Xi = 3$ means $L_a < 0$ and $L_b > 0$; $\Xi = 2$ means $L_a > 0$ and $L_b < 0$; and $\Xi = 1$ means $L_a < 0$ and $L_b < 0$. It is clear that plane waves with both $L_a > 0$ and $L_p < 0$ can have NPV. We notice that only a relatively small proportion of the total $(\theta_k, \phi_k)$ wavevector parameter space supports NPV propagation. This proportion increases as $R$ decreases. The corresponding maps for the $k^+$ NPV wavevectors look just like those presented in Fig. 4 but with $\phi_k$ replaced by $\phi_k + 180^\circ$. 

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Figure 4. The sign of $L_p$ for $k^-$ NPV plane waves mapped against $\theta_k \in [45^\circ, 135^\circ)$ and $\phi_k \in [90^\circ, 180^\circ)$, for neighbourhoods on the equatorial plane specified by $\phi_{KN} = \theta_{KN} = 0$ and $R = 1.51 \, m_{bh}$, 1.75 $m_{bh}$, and 1.9 $m_{bh}$.

Another difference between NPV plane waves in neighbourhoods in the ergosphere and superradiant negative-energy photon production is revealed by a consideration of the frequency/wavelength regimes in which these phenomena occur. The angular frequency of black-hole superradiant photons is bounded, as per $\omega (\omega - m \omega_+) < 0$, where $\omega_+$ is the angular frequency associated with the outer event horizon and $m$ is an integer [40]. In contrast, NPV plane waves in the ergosphere comply with the short-wavelength regimes described in §2.4. However, we emphasize that these short-wavelength regimes are merely sufficient conditions for NPV propagation, arising from the piecewise uniform approximation of $g_{\alpha\beta}$ by $\tilde{g}_{\alpha\beta}$ — the prospects for NPV plane waves at longer wavelengths is a matter for future study.

In conclusion, there are NPV plane waves which are compatible with the negative-energy photon trajectories of superradiance, but there are also NPV plane waves which do not appear to conform to the superradiant scenario.

Parenthetically, we add that a covariant analysis based on $\phi_{KN,KS}$-periodic electromagnetic waves close to the outer event horizon reveals that the superradiant condition $\omega (\omega - m \omega_+) < 0$ follows directly from the covariant equivalent of the NPV condition (10) [43]. This suggests that those NPV plane waves, in neighbourhoods close to the outer event horizon, which do not conform to the superradiant scenario — i.e., those with $L_z > 0$ — may not be superradiant. However, a definitive comparison between plane waves, as considered in the preceding sections, and $\phi_{KN,KS}$-periodic waves necessitates a time-domain study that lies outside the scope of this paper.

5. Concluding remarks

The ergosphere of a rotating black hole is similar to certain negatively refracting metamaterials [1] in flat spacetime, insofar as NPV propagation is supported in both situations. However, unlike the corresponding case for negatively refracting metamaterials, the black-hole NPV condition (10) is satisfied only when the vector
product is evaluated with respect to curved spacetime coordinates. For example, the NPV propagation demonstrated in §3 arises through the implementation of Boyer–Lindquist coordinates, expressed in Kerr–Schild Cartesian form. In a local (inertial) frame of reference, the NPV condition (10) cannot be satisfied for vacuum, in consonance with the Einstein equivalence principle [5].

Our investigation has revealed that the propensity for a rotating black hole to support NPV propagation is highly sensitive to the presence of charge in the black hole. Specifically, (i) the overall incidence of NPV increases as $q_{bh}$ increases, for both the Kerr–Newman and Kerr–Sen metrics; (ii) on a surface of constant $R$, the overall incidence of NPV decreases as $q_{bh}$ increases and the neighbourhoods which support NPV propagation become more concentrated around the ergosphere’s equator; (iii) as $q_{bh}$ increases, differences in the NPV characteristics associated with the Kerr–Newman and Kerr–Sen metrics emerge; and (iv) the greatest differences between the NPV characteristics of the Kerr–Newman and the Kerr–Sen metrics arise close to the outer event horizons of their respective ergospheres. Thus, the Kerr–Newman and Kerr–Sen metric descriptions of a charged rotating black hole may be distinguished on the basis of their NPV characteristics. We conclude by noting that the marked difference in NPV characteristics for the two metric descriptions contrasts with the similarity in their quantum thermal properties [30].

ACKNOWLEDGMENTS

Acknowledgements: BMR is supported by a UK EPSRC grant (GR/S60631/01) and a US NSF Graduate Research Fellowship. TGM and AL thank Dr. Sandi Setiawan (University of Edinburgh) for suggesting this study and Prof. Malcolm MacCallum (Queen Mary, University of London) for his assistance with §2.3.

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