*In this worksheet, assume all vectors are normalized, i.e., $\mathbf{v}^\dagger \mathbf{v} = 1$

1 Hermitian and unitary matrices

(a) Show that the eigenvalues of a Hermitian matrix are real.

(b) Using the fact that the eigenvalues of a Hermitian matrix H are real, show that if the eigenvalues are nondegenerate, the corresponding eigenvectors are orthogonal, i.e., show that

$$\mathbf{v}_m^\dagger \mathbf{v}_n = \delta_{mn} \qquad \text{if} \qquad H\mathbf{v}_n = \epsilon_n \mathbf{v}_n, \text{ and } \epsilon_m \neq \epsilon_n \text{ for } m \neq n \tag{1}$$

where we are assuming that each $\mathbf{v}_n$ is normalized.

(c) Let U be a square matrix whose columns are the normalized eigenvectors of a Hermitian matrix, and let I be the identity matrix so that

$$I_{mn} = \delta_{mn} \qquad \text{and} \qquad I\mathbf{v} = \mathbf{v} \tag{2}$$

for any vector $\mathbf{v}$. Show that

$$U^\dagger U = UU^\dagger = I. \tag{3}$$

Note: by definition, all unitary operators obey equation **??**!

2 Functions of matrices

(a) Consider the diagonal matrix

$$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{4}$$

where a and b are real numbers. What is D^n?

(b) What is $f(D)$ in terms of $f(a)$ and $f(b)$?

(c) Consider a 2×2 Hermitian matrix A with eigenvalues a_1 and a_2. Let U be a 2×2 matrix whose columns are the normalized eigenvectors of A. Show that the diagonal matrix

$$D = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \tag{5}$$

can be written as some product involving A and U.

(d) Matrix U has a special property. State the property, and prove that U has it. Hint: This matrix property was used on the previous page.

(e) Use equation **??** to write is A in terms of D and U.

(f) What is A^n in terms of D and U? Use equation **??** to simplify the expression as much as possible.

(g) Show that the function $f(A)$ can be written as

$$f(A) = U \begin{pmatrix} f(a_1) & 0 \\ 0 & f(a_2) \end{pmatrix} U^\dagger \tag{6}$$

3 Application

The goal of this section is to find the matrix representation of the operator $\hat{R}(\theta, \mathbf{n}) = e^{-i\theta\hat{\mathbf{J}}\cdot\mathbf{n}/\hbar}$, which rotates an arbitrary spin-1/2 state by an angle θ about an axis $\mathbf{n} = (n_x, n_y, n_z)$, letting $|\mathbf{n}|^2 = 1$. Here, the components of $\hat{\mathbf{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$ are the total angular momentum operators in the x, y, and z directions. Their matrix representations in the $|\pm z\rangle$ basis are given in the text as

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

(a) We will now apply the methods we worked out in parts 2??-2??. Write the 2×2 matrix representation of the operator $\hat{\mathbf{J}} \cdot \mathbf{n}$ in the $|\pm z\rangle$ basis in terms of n_x, n_y, and n_z. We'll call this matrix $\mathbf{J} \cdot \mathbf{n}$ (removing the "hat").

(b) Find the eigenvalues and normalized eigenvectors of $\mathbf{J} \cdot \mathbf{n}$.

(c) Write $\mathbf{J} \cdot \mathbf{n}$ as the product of a diagonal matrix and two unitary matrices.

(d) Write $R(\theta, \mathbf{n}) = e^{-i\theta\mathbf{J}\cdot\mathbf{n}/\hbar}$, the matrix repesentation of $\hat{R}(\theta, \mathbf{n})$, as the product of a diagonal matrix and two unitary matrices.

(e) Carry out the matrix product you arrived at in the previous question. Show that the resulting matrix can be written as

$$R(\theta, \mathbf{n}) = I \cos\left(\frac{\theta}{2}\right) - \frac{2i}{\hbar}(\mathbf{J} \cdot \mathbf{n}) \sin\left(\frac{\theta}{2}\right), \tag{8}$$

where I is the identity matrix.

4 Reflection

(a) Check that the matrix $R(\theta, \mathbf{n})$ correctly rotates the $|+z\rangle$ state into $|-z\rangle$, $|+x\rangle$, and $|-x\rangle$.

(b) In a bulleted list, summarize all of the important steps that allowed us to write the matrix representation of the function of an operator.

5 Outer Products

Since $|+z\rangle$ and $|-z\rangle$ form a complete basis, any matrix can be formed out of a linear combination of direct products of the basis bras and kets.

Recall $|+z\rangle$ and $|-z\rangle$ may be represented in the S_z basis as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively. It can also be seen that the outer product $|+z\rangle\langle+z|$ may be represented as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{9}$$

and the outer product $|+z\rangle\langle-z|$ may be represented as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{10}$$

(a) Using outer products, find a way to form the Identity matrix out of outer products of the $|+z\rangle$ and $|-z\rangle$ bras and kets in the S_z basis.

(b) Similarly, form the Pauli matrix σ_z.

(c) If you are up for a little challenge, form the Pauli σ_y, and σ_x matrices out of a linear combination of outer products of $|+z\rangle$ and $|-z\rangle$ bras and kets in the S_z basis.