Dr. Vincent Meunier, April 15th, 2021

This is a guided step-by-step approach to solve a central potential problem. This is a HW assignment. Submit your work on gradescope.

The differential equation for the radial wave function of the isotropic three-dimensional harmonic oscillator is

$$\frac{d^2u}{dr^2} - \frac{l(l+1)}{r^2}u - r^2u = -\lambda u, \tag{1}$$

where r is a dimensionless measure of the distance, l is related to the angular momentum, and $\lambda = \frac{2E}{\hbar\omega}$ is the dimensionless energy.

1. Large r behavior.
   (a) Show that for $r \to \infty$, equation 1 reduces to
   $$\frac{d^2u}{dr^2} = r^2u \tag{2}$$
   (b) Using equation 2, find the large r behavior of $u(r)$.
2. Small r behavior.
   (a) Show that for $r \to 0$, the equation reduces to
   $$\frac{d^2u}{dr^2} = \frac{l(l+1)}{r^2}u \tag{3}$$
   (b) Based on equation 3, try a solution $u(r \to 0) = r^s$ and determine the possible values of s. Show why one value should be discarded.
3. If you got everything right thus far, your work should indicate that a possible solution to Eq. 1 is $u(r) = r^{l+1}e^{-r^2/2}F(r)$. We will now seek a series solution for $F(r)$.

   (a) Calculate $\frac{d^2u}{dr^2}$. This will yield a differential equation for $F(r)$. Do not substitute just yet.
   (b) Use this result to evaluate Eq. 1. Try a series solution $F(r) = \sum_{k=0}^{\infty} c_k r^k$.
   (c) If everything is right, you should have found a recursion relation between c_k and c_{k+2}. Show why this recursion relation indicates that the series must terminate at a finite value of k. For this, calculate $\lim_{k\to\infty} \frac{c_{k+2}}{c_k}$ and show that this limit corresponds to an unacceptable solution.
   (d) Finally, show that imposing the termination of the series yields a quantization of the energy.

   You have now solved the three-dimensional radial equation of the isotropic harmonic oscillator and you should be very happy.