1. Normalize the following vectors ($i = \sqrt{-1}$).

   i. $\mathbf{v} = 3\hat{\mathbf{x}} - 4\hat{\mathbf{z}}$

   ii. $\mathbf{w} = -2\hat{\mathbf{x}} + i\hat{\mathbf{y}} + \sqrt{3}\hat{\mathbf{z}}$

2. Calculate the matrix-matrix and matrix-vector products shown below.

   i. $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}$

   ii. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

3. Invert the following matrices. If a matrix has no inverse, explain why.

   i. $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

   ii. $B = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$

4. Solve the following matrix equations. If an equation has no solution, explain why.

   i. $\begin{pmatrix} 3 & 5 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -11 \end{pmatrix}$

   ii. $\begin{pmatrix} 1 & -2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

5. Calculate the inner ($\mathbf{u} \cdot \mathbf{v}$) and cross ($\mathbf{u} \times \mathbf{v}$) products of the following pairs of vectors.

i. $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

ii. $\mathbf{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$, $\mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$

6. Calculate the eigenvalues and eigenvectors for the following matrices.

i. $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

ii. $B = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

7. What can you say about the eigenvalues and eigenvectors of a Hermitian matrix?

8. If a matrix is unitary, what equation does it satisfy?

9. Determine whether or not these sets of vectors form a basis (hint: there are two requirements).

   i. $V = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right\}$ ii. $U = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -4 \end{pmatrix} \right\}$

10. For the two bases shown below

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad , \quad B' = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

   determine the transition matrices to go from

   i B' to B ii. B to B'

11. Using the results of the previous problem, find the representations of the following vectors (initially expressed in basis B) in basis B'.

   i $\mathbf{v} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}_B$ ii. $\mathbf{v} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}_B$

12. Find the representation of the matrix $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}_B$ in the basis B'.

13. Give the first four Taylor series terms for the following functions. For each function, expand about whatever value of x you see fit.

    i. $f(x) = \sin(x)$ ii. $g(x) = e^x$ iii. $h(x) = (1 + x)^n$ for non-integer n

14. Find the Fourier transform of the function $f(x) = \sin(x) + 2\cos(3x)$.

We will now go over how to take the determinant of 4×4 and $n \times n$ matricies. This process can be broken down into steps. Let us do this with the following matrix:

$$A = \begin{pmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{pmatrix} \tag{1}$$

1. Pick a row or column. It does not matter which row or column we pick, the determinant will be unique regardless of this. We can pick such a row or column strategically, as we will see. Let us select the highlighted column.

$$\begin{pmatrix} 5 & -7 & 2 & 2 \\ 1 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 1 & 5 & 0 & -6 \end{pmatrix} \tag{2}$$

2. Find the sub-matricies and their determinants. For each component in the highlighted column, we cover up its row and column to find a 3×3 matrix. For example, we do this with the top component -7:

$$\begin{pmatrix} 5 & -7 & 2 & 2 \\ 1 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 1 & 5 & 0 & -6 \end{pmatrix} \tag{3}$$

Omitting these values, we find the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & -4 \\ -5 & 0 & 3 \\ 1 & 0 & -6 \end{pmatrix} \tag{4}$$

Its determinant is

$$\begin{vmatrix} 1 & 0 & -4 \\ -5 & 0 & 3 \\ 1 & 0 & -6 \end{vmatrix} = 0 \tag{5}$$

Following with the other components, we go down the column and get the matricies

$$\begin{pmatrix} 5 & -7 & 2 & 2 \\ 1 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 1 & 5 & 0 & -6 \end{pmatrix} \rightarrow \begin{vmatrix} 5 & 2 & 2 \\ -5 & 0 & 3 \\ 1 & 0 & -6 \end{vmatrix} = -54 \tag{6}$$

$$\begin{pmatrix} 5 & -7 & 2 & 2 \\ 1 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 1 & 5 & 0 & -6 \end{pmatrix} \rightarrow \begin{vmatrix} 5 & 2 & 2 \\ 1 & 0 & -4 \\ 1 & 0 & -6 \end{vmatrix} = 4 \tag{7}$$

$$\begin{pmatrix} 5 & -7 & 2 & 2 \\ 1 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 1 & 5 & 0 & -6 \end{pmatrix} \rightarrow \begin{vmatrix} 5 & 2 & 2 \\ 1 & 0 & -4 \\ -5 & 0 & 3 \end{vmatrix} = 34 \tag{8}$$

3. Now we find what is known as the 'cofactor' for each of our components. This is simply equal to $C_{i,j} = (-1)^{i+j}$ where i and j are the position of the component in the matrix. For example, for -7, the cofactor would be $C_{1,2} = (-1)^{1+2} = -1$.

4. We combine all of these in a sum:

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 1 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 1 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+2}(0)(-7) + (-1)^{2+2}(-54)(3) + (-1)^{3+2}(4)(-8) + (-1)^{4+2}(34)(5) \quad (9)$$

Simplifying,

$$\det(A) = -1(0)(-7) + 1(-54)(3) + -1(4)(-8) + 1(34)(5) = 40 \quad (10)$$

Thus we have our determinant. One may have noticed, that this problem would be easier if we selected column number three:

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 1 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 1 & 5 & 0 & -6 \end{vmatrix} \quad (11)$$

Try this on your own.

Additional Problems:

1. Find the determinants:

i $A = \begin{pmatrix} 5 & -7 & 2 & 2 \\ 1 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 1 & 5 & 0 & -6 \end{pmatrix}$

ii. $B = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$

2. Find all eigenvalues (not eigenvectors!) of the following matricies:

i $C = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}$

ii. $D = -i \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & -1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}$