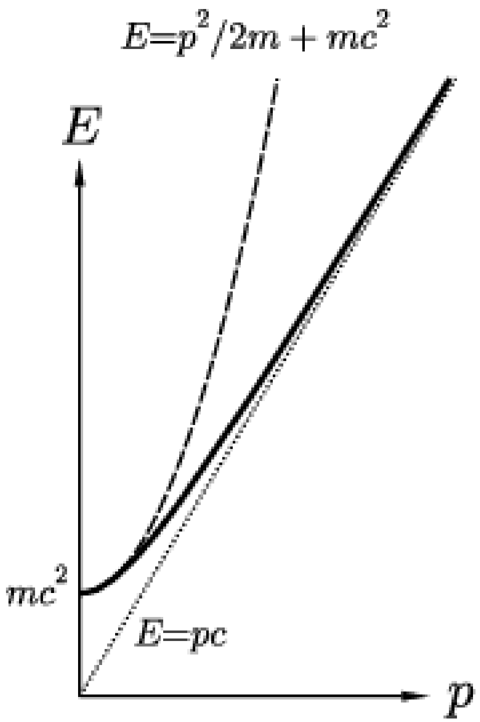
Lecture 25. *Relativistic Gas*

- In this lecture, we will focus on cases where classical mechanics description of kinetic energy fails at high velocity

Dispersion relation



Kinetic Energy, relativistic formula

$$E^2 = p^2c^2 + m^2c^4$$

m: rest mass

Non-relativistic limit

$p/m \ll c$ ➡ $E = \dfrac{p^2}{2m} + mc^2$

mc^2: rest mass energy

$E(k) = \hbar^2 k^2/2m$

Ultra-relativistic limit

$p/m \gg c$ ➡ $E = pc$

e.g., photons

$\omega = ck$

Ultrarelativistic gas $E = pc$

We will consider an UR gas of finite mass particles

Partition function of a single-particle: $Z_1 = \int_0^\infty e^{-\beta\hbar kc} g(k)\, dk$

Density of states: $g(k)\, dk = \frac{V k^2\, dk}{2\pi^2}$

$$\longrightarrow \quad Z_1 = \frac{V}{2\pi^2}\left(\frac{1}{\beta\hbar c}\right)^3 \int_0^\infty e^{-x} x^2\, dx \qquad x = \beta\hbar kc$$

$$\longrightarrow \quad Z_1 = \frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3$$

$$Z_1 = \frac{V}{\Lambda^3} \qquad \Lambda = \frac{c\pi^{2/3}}{k_B T}$$

What changed? T^{-3} instead of $T^{-3/2}$

Thermodynamic properties!

- Partition function of the ultra relativistic gas of *indistinguishable* particles

$$Z_N = \frac{Z_1^N}{N!}$$

*note: when is this valid?

$$\ln Z_N = N \ln V + 3N \ln T + \text{constants}$$

- Internal energy $$U = -\frac{\mathrm{d} \ln Z_N}{\mathrm{d}\beta} = 3Nk_{\mathrm{B}}T.$$

- Heat capacity $$C_V = \left(\frac{\partial U}{\partial T}\right)_V \rightarrow C_V = 3Nk_{\mathrm{B}}$$

*note: why is this different from the equipartition theorem?

$$\ln Z_N = N \ln V + 3N \ln T + \text{constants}$$

- **Internal energy** $U = -\frac{\mathrm{d} \ln Z_N}{\mathrm{d}\beta} = 3Nk_\mathrm{B}T$

- **Heat capacity** $C_V = \left(\frac{\partial U}{\partial T}\right)_V \longrightarrow C_V = 3Nk_\mathrm{B}$

- **Helmholtz function** $F = -k_\mathrm{B}T \ln Z_N = -k_\mathrm{B}TN \ln V - 3Nk_\mathrm{B}T \ln T - k_\mathrm{B}T \times \text{constants}$

- **Pressure** $p = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{Nk_\mathrm{B}T}{V} = nk_\mathrm{B}T$ *ideal gas law!

- **Enthalpy** $H = U + pV = 4Nk_\mathrm{B}T$

Entropy

$$\Lambda = \frac{c\pi^{2/3}}{k_{\rm B}T}$$

$$\begin{aligned} \ln Z_N &= N\ln V - 3N\ln\Lambda - N\ln N + N \\ &= N\ln\left(\frac{1}{n\Lambda^3}\right) + N \end{aligned}$$

$$\begin{aligned} F &= -k_{\rm B}T\ln Z_N \\ &= Nk_{\rm B}T[\ln(n\Lambda^3) - 1], \\ S &= \frac{U-F}{T} \\ &= Nk_{\rm B}[4 - \ln(n\Lambda^3)], \\ G &= H - TS = 4Nk_{\rm B}T - Nk_{\rm B}T[4 - \ln(n\Lambda^3)] \\ &= Nk_{\rm B}T\ln(n\Lambda^3). \end{aligned}$$

Pressure

$$p = \frac{u}{3}$$

$$p = \frac{2u}{3}$$

Adiabatic Expansion of an ultra relativistic gas

- No heat in or out!
- No entropy change $S = Nk_{\rm B}[4 - \ln(n\Lambda^3)]$
- Thus this is also a constant $n\Lambda^3 = n(\frac{\hbar c\pi^{2/3}}{k_{\rm B}T})^3 = \frac{N}{V}(\frac{\hbar c\pi^{2/3}}{k_{\rm B}T})^3$

Expansion of the universe

$$VT^3 = \text{constant}$$

$$T \propto V^{-1/3} \propto a^{-1}$$

- a is a scale factor

- The temperature of the CMB is inversely proportional to the scale factor of the universe.

- Non-relativistic gas: $VT^{3/2} = \text{constant}$ $\quad T \propto V^{-2/3} \propto a^{-2}$.

- **Faster cooling than UR**

Density of the expanding universe

- Non-relativistic gas: $\rho \propto V^{-1} \longrightarrow \rho \propto a^{-3}$

- Relativistic gas: $\rho = \frac{u}{c^2} \longrightarrow \begin{matrix} u = 3p \\ p \propto V^{-4/3} \end{matrix} \longrightarrow \rho \propto a^{-4}$

The density of relativistic particles decreases faster than that of non-relativistic particles, as the universe expands and temperature goes down

Early universe was radiation dominated and the Universe has become matter dominated

Summary

Property	Non-relativistic	Ultrarelativistic
Z_1	$\dfrac{V}{\lambda_{\text{th}}^3}$ $\lambda_{\text{th}} = \dfrac{h}{\sqrt{2\pi m k_B T}}$	$\dfrac{V}{\Lambda^3}$ $\Lambda = \dfrac{c\pi^{2/3}}{k_B T}$
U	$\frac{3}{2}Nk_BT$	$3Nk_BT$
H	$\frac{5}{2}Nk_BT$	$4Nk_BT$
p	$\dfrac{Nk_BT}{V}$ $= \dfrac{2u}{3}$	$\dfrac{Nk_BT}{V}$ $= \dfrac{u}{3}$
F	$Nk_BT[\ln(n\lambda_{\text{th}}^3) - 1]$	$Nk_BT[\ln(n\Lambda^3) - 1]$
S	$Nk_B[\frac{5}{2} - \ln(n\lambda_{\text{th}}^3)]$	$Nk_B[4 - \ln(n\Lambda^3)]$
G	$Nk_BT \ln(n\lambda_{\text{th}}^3)$	$Nk_BT \ln(n\Lambda^3)$
Adiabatic expansion	$VT^{3/2} = \text{constant}$ $pV^{5/3} = \text{constant}$	$VT^3 = \text{constant}$ $pV^{4/3} = \text{constant}$