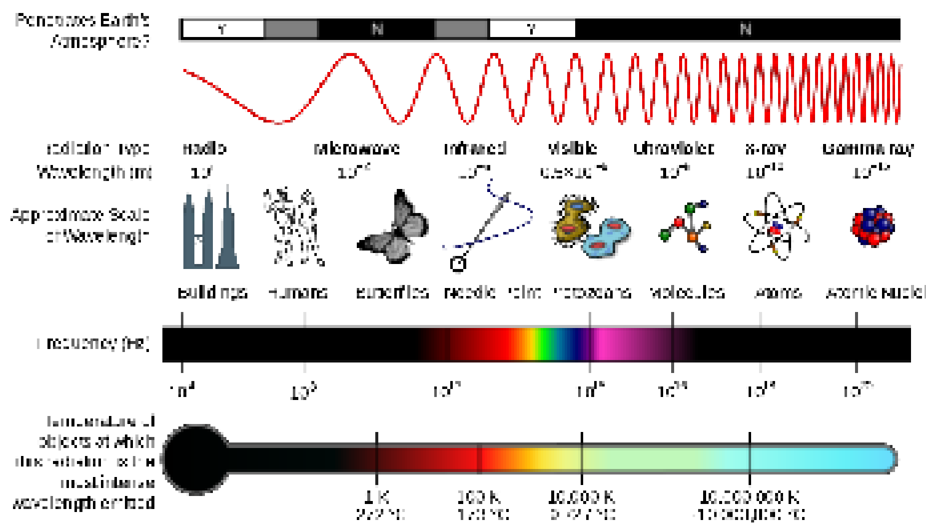
Lecture 23. *Photons*

- In this lecture, we will focus on the thermodynamics of electromagnetic radiation

$$\frac{\omega}{k} = 2\pi\nu \times \frac{\lambda}{2\pi} = \nu\lambda = c$$

Planck: quantized energy



https://en.wikiversity.org/wiki/Electromagnetic_radiation

Thermal radiation

Any substance emits EM radiation at T>0

- In the infrared regime (low-frequency) at room temperature
- If T is really large: you can actually see the thermal radiation with the naked eye

Pomeranian (wikipedia)

https://hotmetalworks.squarespace.com/

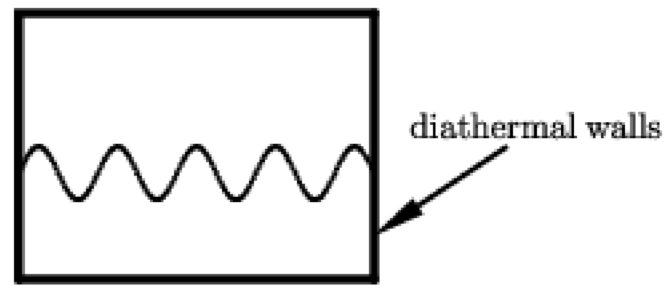
Classical thermodynamics

Cavity of volume V:

Surroundings containing a collection of photons

Photons in thermal equilibrium with the cavity walls

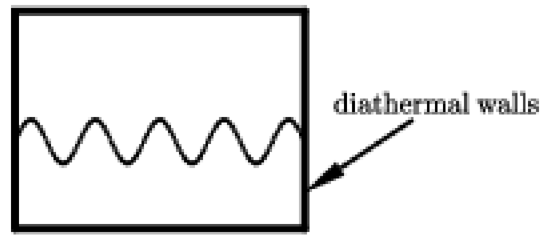
Photons form EM standing waves



Diathermal material: transmits heat between the gas of photons and the surroundings

Classical thermodynamics

Imagine n photons per unit volume in the cavity



Energy density (using mean energy): $u = \frac{U}{V} = n\hbar\omega$

Recall: for a gas of particle, the pressure is $\frac{1}{3}nm\langle v^2\rangle$ ⟶ $p = \frac{u}{3}$

"radiation pressure"

Flux of photons on the walls*:
(# of photons striking the walls per second per unit area): $\Phi = \frac{1}{4}nc$

*chapter 7

Power incident per unit area

Flux of photons on the walls*:
(# of photons striking the walls per second per unit area): $\Phi = \frac{1}{4} nc$

*chapter 7

Incident power: $$\boxed{F = \hbar\omega\Phi = \frac{1}{4}uc}$$

Stefan-Boltzmann law

- What is the energy flux radiating from a body at temperature T?
- First law of thermodynamics: $\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V$

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p$$

$$= T\left(\frac{\partial p}{\partial T}\right)_V - p$$

(Maxwell relation)

$$= \frac{1}{3}T\left(\frac{\partial u}{\partial T}\right)_V - \frac{u}{3}$$

Left-hand side

$$U = uV$$

$$\left(\frac{\partial U}{\partial V}\right)_T = u + V\left(\frac{\partial u}{\partial V}\right)_T = u$$

$$u = \frac{1}{3}T\left(\frac{\partial u}{\partial T}\right)_V - \frac{u}{3}$$

$$\Rightarrow \quad 4u = T\left(\frac{\partial u}{\partial T}\right)_V$$

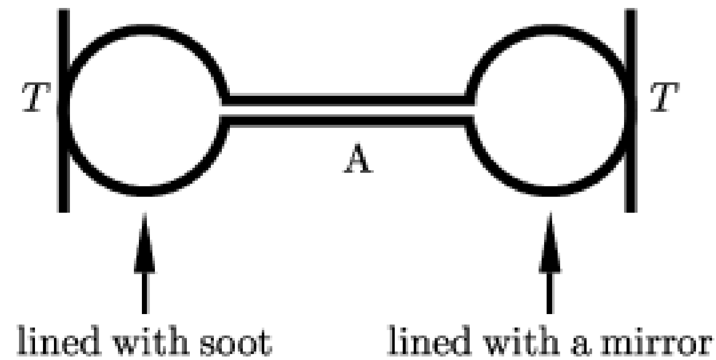
$$F = \sigma T^4$$

Incident power per unit area (energy flux, radiative flux, irradiance)

Spectral energy density

$$F = \quad \omega\Phi = \frac{1}{4}uc$$

- **Energy density of EM radiatio**n = how many Joules are stored in a cubic meter of cavity
- What is the frequency range of that storage?



Same T -> no heat exchange

Same flux out of left and right!

Thus both cavities must have same energy density (u)

Because flux only depends on density (not on coating, shape of value), we conclude that u is independent of shape, size, or material of the cavity

This is a **weak** argument since we only know for "u", not the specific wavelengths

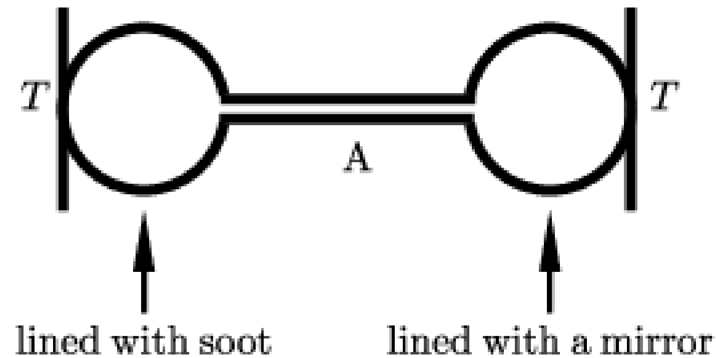
Spectral energy density

- Spectral energy density: $u_\lambda d\lambda$ is the energy density due to this photons which have wavelengths between λ and $\lambda + d\lambda$
- If we place a frequency filter at A on the figure, we see that

$$u_\lambda^{\text{soot}}(T) = u_\lambda^{\text{mirror}}(T)$$

We conclude that the spectral internal energy has no dependence on the material, shape, size, or nature of a cavity; it is thus a universal function of T and λ

$$u = \int u_\lambda \, d\lambda$$



Kirchhoff's law

- How well does a particular surface absorb or emit EM radiation at a given frequency?

- **The spectral absorptivity** α_λ is the fraction [dimensionless] of the incident radiation that is absorbed at wavelength λ
- **The spectral emissive power** e_λ of a surface is a function such that $e_\lambda \mathrm{d}\lambda$ is the power emitted per unit area by the electromagnetic radiation having wavelengths between λ and $\lambda + \mathrm{d}\lambda$

Incident spectral energy density: $u_\lambda \mathrm{d}\lambda$

Power per unit area absorbed: $\left(\frac{1}{4} u_\lambda \mathrm{d}\lambda\, c\right) \alpha_\lambda$

Power per unit area emitted: $e_\lambda \mathrm{d}\lambda$

At equilibrium:

$$\frac{e_\lambda}{\alpha_\lambda} = \frac{c}{4} u_\lambda$$

Equilibrium:

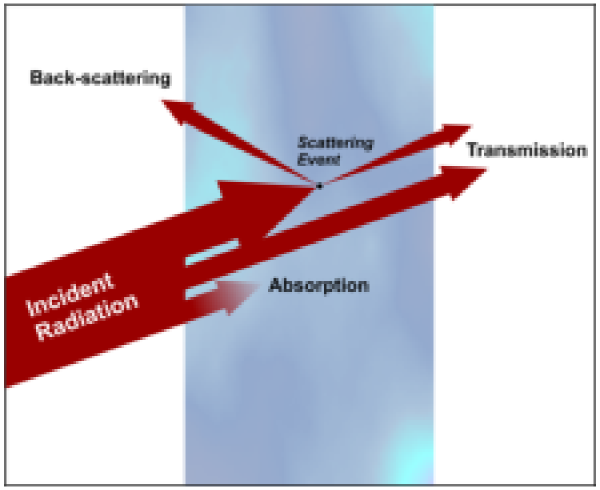
$$\frac{e_\lambda}{\alpha_\lambda} = \frac{c}{4} u_\lambda$$

The ratio is a universal function of λ and T

"Good (bad) absorbers are good (bad) emitters"

Perfect black body

- Perfect absorber at all wavelength ($\alpha_\lambda = 1$)
- Therefore it is also the best emitter (according to Kirchhoff's law)
- **Black body cavity**: a cavity made up with black-body walls. It contains photons at same T as the walls, due to emission and absorption of photons by the atoms of the walls
- The gas of phonon the the cavity is known as the black-body radiation

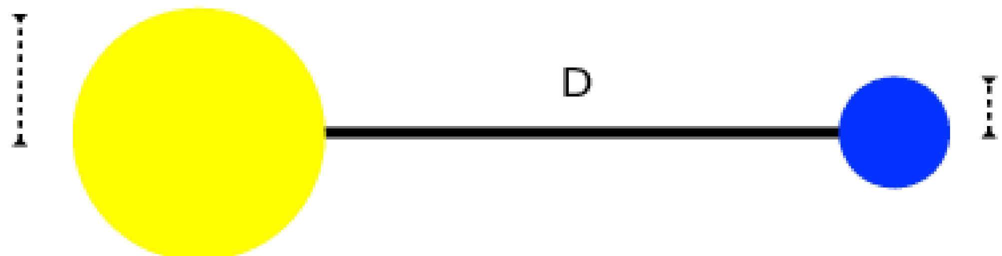


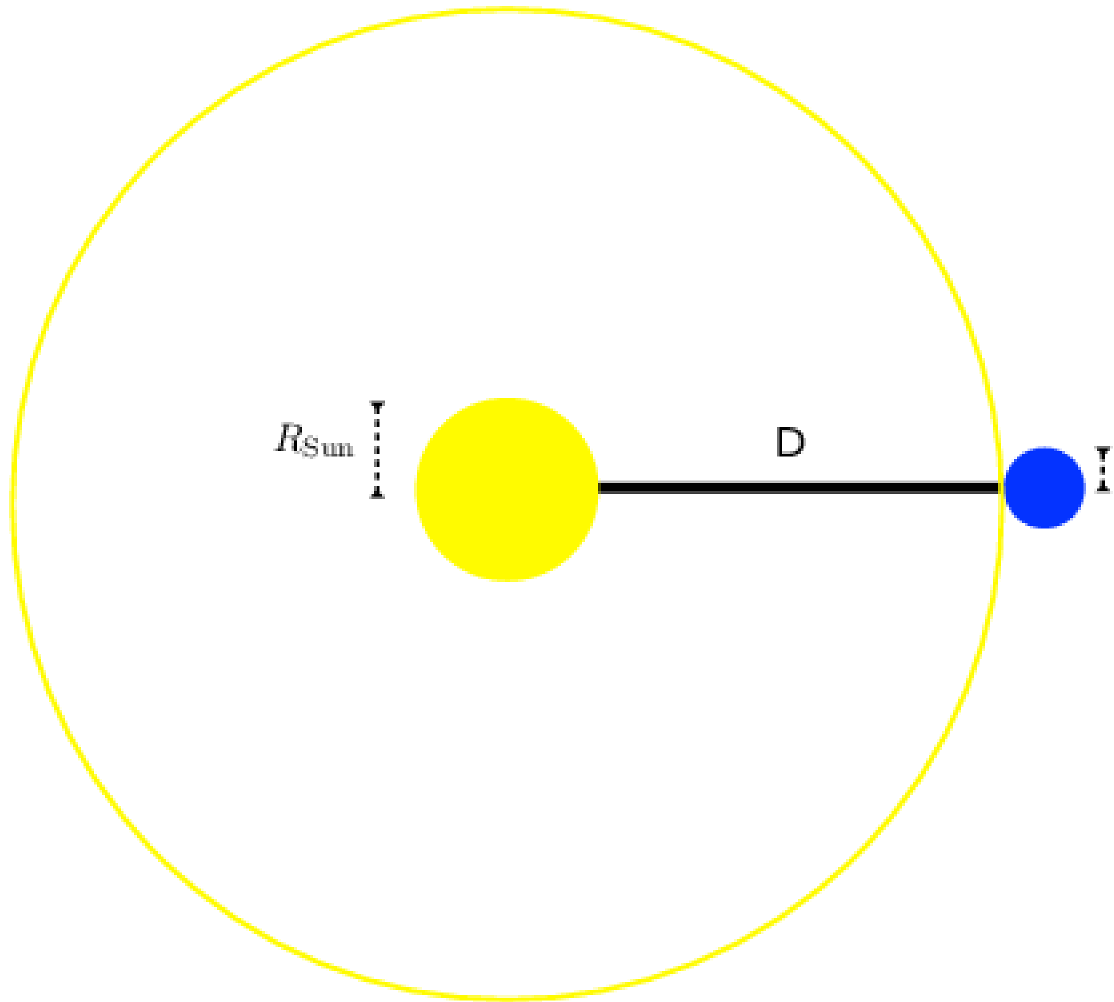
https://www.e-education.psu.edu/eme810/node/476

Application: temperature of Earth

- Imagine that Sun and Earth behave like black-bodies
- Luminosity: power per surface area (Stefan) x surface area

$$L = 4\pi R_{\text{Sun}}^2 \sigma T_{\text{Sun}}^4$$





$$L = 4\pi R_{\text{Sun}}^2 \sigma T_{\text{Sun}}^4$$

Incident power: $L\left(\frac{\pi R_{\text{Earth}}^2}{4\pi D^2}\right)$

Power emitted: $4\pi R_{\text{Earth}}^2 \sigma T_{\text{Earth}}^4$

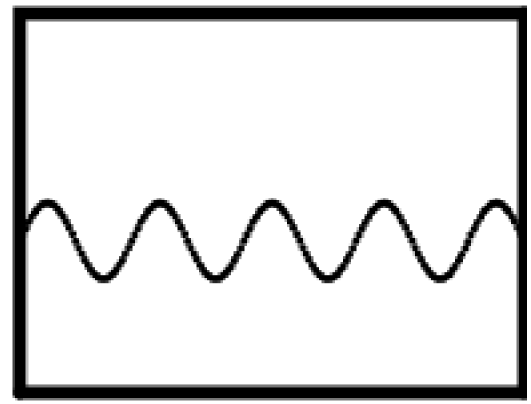
$$\frac{T_{\text{Earth}}}{T_{\text{Sun}}} = \sqrt{\frac{R_{\text{Sun}}}{2D}}$$

Putting in the numbers
$R_{\text{Sun}} = 7 \times 10^8 \text{m}, D = 1.5 \times 10^{11} \text{m},$ and

$T_{\text{Sun}} = 5800\text{K}$ yields $T_{\text{Earth}} = 280\text{K}$, which is not bad given the crudeness of the assumptions.

Summary so far for black-body radiation

$$\begin{aligned}
\text{power radiated per unit area} \quad F &= \frac{1}{4}uc = \sigma T^4, \\
\text{energy density in radiation} \quad u &= \left(\frac{4\sigma}{c}\right) T^4, \\
\text{pressure on cavity walls} \quad p &= \frac{u}{3} = \frac{4\sigma T^4}{3c}.
\end{aligned}$$



What about the behavior for a beam of light?

A cubic meter of a collimated beam of light has momentum:

$$n\hbar k = n\hbar\omega/c$$

This momentum is absorbed by unit area of surface normal to the beam in a time 1/c and the pressure is:

$$p = [n\hbar\omega/c]/[1/c] = n\hbar\omega = u$$

Power per unit area of surface is thus: $F = n\hbar\omega/(1/c) = uc.$

$$\begin{aligned} \text{power radiated per unit area} \quad F &= uc = \sigma T^4 \\ \text{energy density in radiation} \quad u &= \left(\frac{\sigma}{c}\right) T^4, \\ \text{pressure on cavity walls} \quad p &= u = \frac{\sigma T^4}{c}. \end{aligned}$$

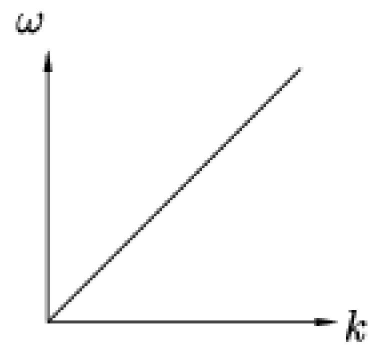
Pressure of radiation?

- Sunlight on earth: $F = 1370 W/m^2$
- Thus the pressure is: $p = \frac{F}{c} = 4.6\,\mu\mathrm{Pa}$ (all going in the same direction)

This is, of course, tiny compared to atmospheric pressure

Statistical mechanics treatment

- An electromagnetic wave in a cavity can be described as a simple harmonic oscillator, of angular frequency ω. The momentum of each mode is k; with $\omega = ck$.



- The density of states is (similar to what we did for the ideal gas and for a cubic cavity of size L):

$$g(k)\,\mathrm{d}k = \frac{4\pi k^2\,\mathrm{d}k}{(2\pi/L)^3} \times 2$$ ← 2 types of polarization

- That is:

$$g(k)\,\mathrm{d}k = \frac{Vk^2\,\mathrm{d}k}{\pi^2}$$

$$g(k)\,\mathrm{d}k = \frac{Vk^2\,\mathrm{d}k}{\pi^2} \longrightarrow g(\omega)$$

$$g(\omega) = g(k)\,\frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{g(k)}{c} \longrightarrow g(\omega)\,\mathrm{d}\omega = \frac{V\omega^2\,\mathrm{d}\omega}{\pi^2 c^3}$$

Energy: $U = \int_0^\infty g(\omega)\,\mathrm{d}\omega\,\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1}\right)$ Equation for single harmonic oscillator

$$U = \int_0^\infty g(\omega)\,\mathrm{d}\omega\,\hbar\omega\left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1}\right)$$

$$\int_0^\infty g(\omega)\,\mathrm{d}\omega\,\frac{1}{2}\hbar\omega \to \infty$$ → Energy of vacuum

$$U = \int_0^\infty g(\omega)\,\mathrm{d}\omega\,\frac{\hbar\omega}{\mathrm{e}^{\beta\hbar\omega} - 1} = \frac{V\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3\,\mathrm{d}\omega}{\mathrm{e}^{\beta\hbar\omega} - 1}.$$ → $$U = \frac{V\hbar}{\pi^2 c^3}\left(\frac{1}{\hbar\beta}\right)^4 \int_0^\infty \frac{x^3\,\mathrm{d}x}{\mathrm{e}^x - 1} = \left(\frac{V\pi^2 k_\mathrm{B}^4}{15c^3\hbar^3}\right)T^4.$$

Internal energy of oscillator

$$\int_0^\infty \frac{x^3\,\mathrm{d}x}{\mathrm{e}^x - 1} = \zeta(4)\Gamma(4) = \frac{\pi^4}{15}$$

→ $$A = \frac{\pi^2 k_\mathrm{B}^4}{15c^3\hbar^3}$$

$$A = 4\sigma/c$$ → $$\sigma = \frac{\pi^2 k_\mathrm{B}^4}{60c^2\hbar^3} = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$$

Stefan-Botzmann constant

Black-body distribution

$$U = \int_0^\infty g(\omega)\,\mathrm{d}\omega\,\hbar\omega\left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1}\right)$$

$$u = \frac{U}{V} = \int u_\omega\,\mathrm{d}\omega$$

Energy density

Black body distribution

$$u_\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{\mathrm{e}^{\beta\hbar\omega} - 1}$$

Spectral energy density

In terms of frequency: $u_\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{\mathrm{e}^{\beta h\nu} - 1}$

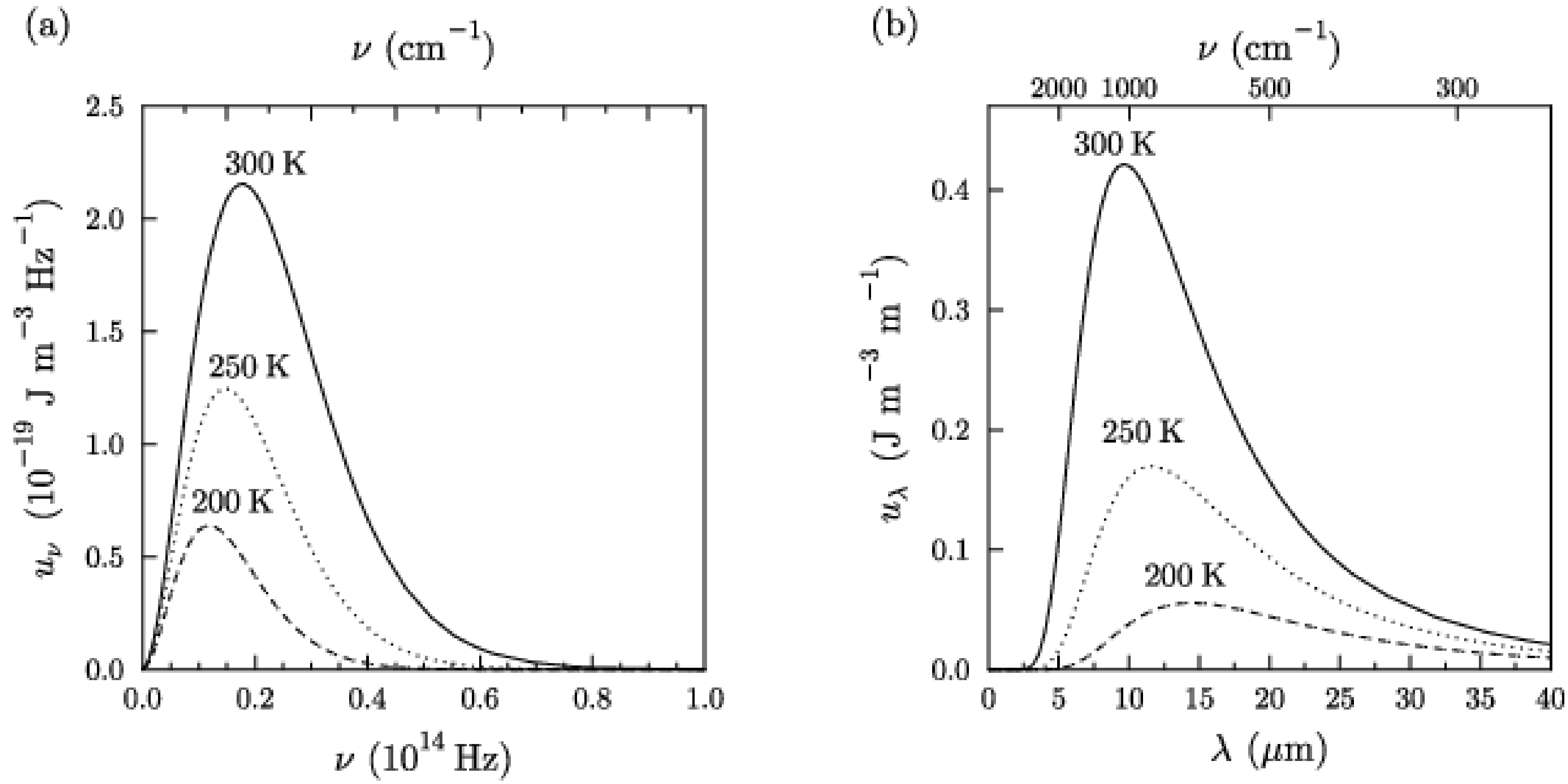
In terms of wavelength:

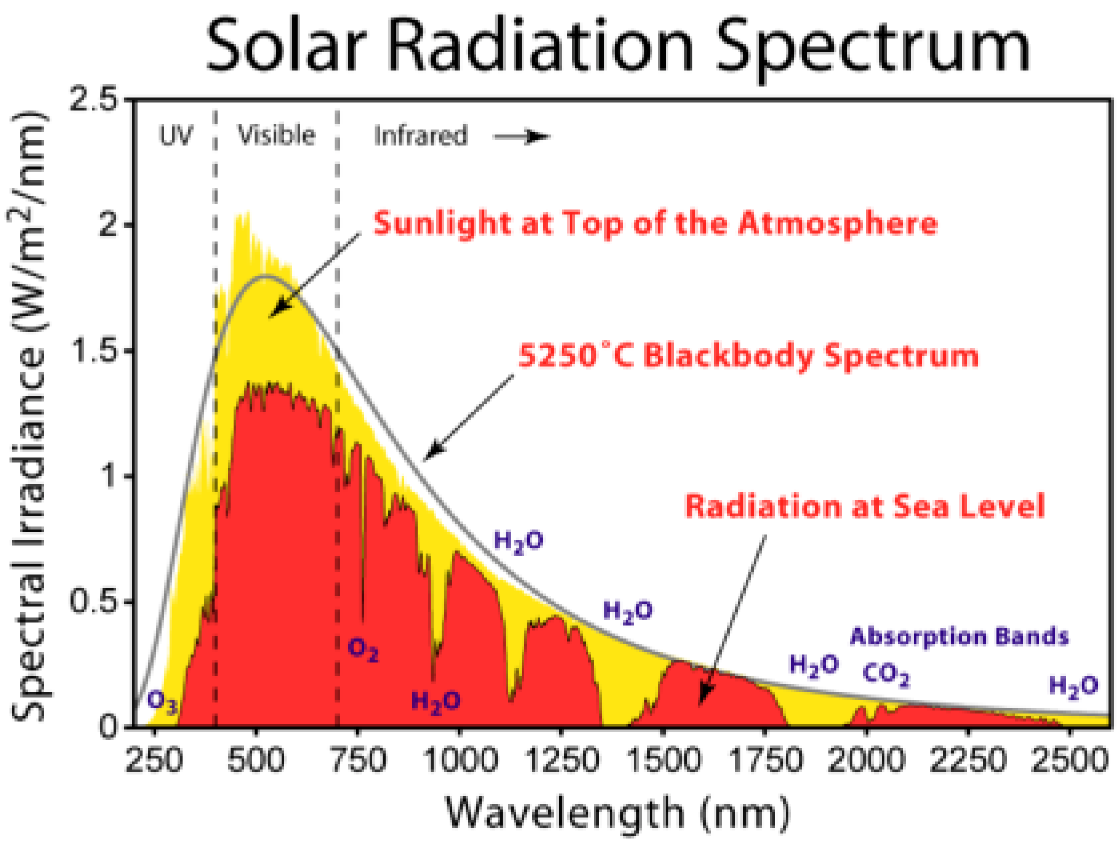
$$\nu = c/\lambda$$

$$\mathrm{d}\nu/\mathrm{d}\lambda = -c/\lambda^2$$

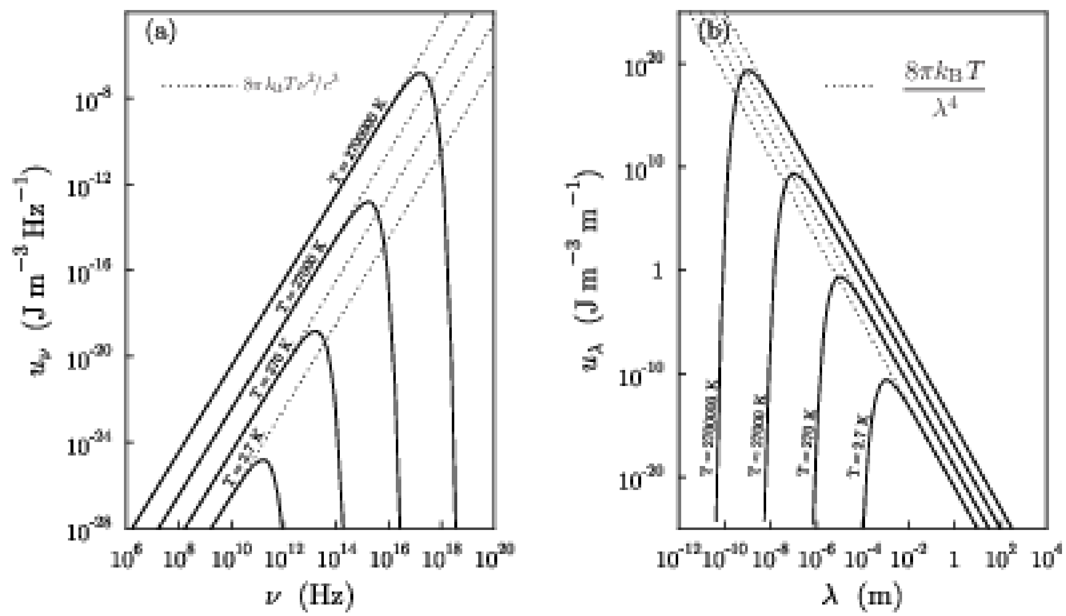
$$u_\nu\,|\mathrm{d}\nu| = u_\lambda\,|\mathrm{d}\lambda|$$

$$u_\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{\mathrm{e}^{\beta hc/\lambda} - 1}$$


(a)
ν (cm⁻¹)
300 K
250 K
200 K
u_ν (10⁻¹⁹ J m⁻³ Hz⁻¹)
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
ν (10¹⁴ Hz)
(b)
ν (cm⁻¹)
2000 1000
500
300
300 K
250 K
200 K
u_λ (J m⁻³ m⁻¹)
0.4
0.3
0.2
0.1
0.0
0
5
10
15
20
25
30
35
40
λ (μm)



https://scipol.org/learn/science-library/solar-power-0



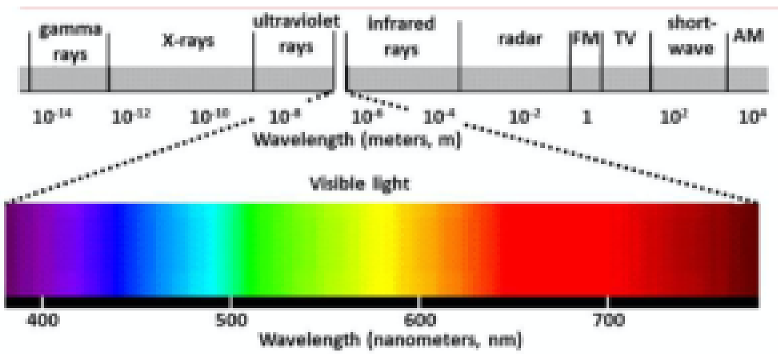
Low frequency (long wavelength)

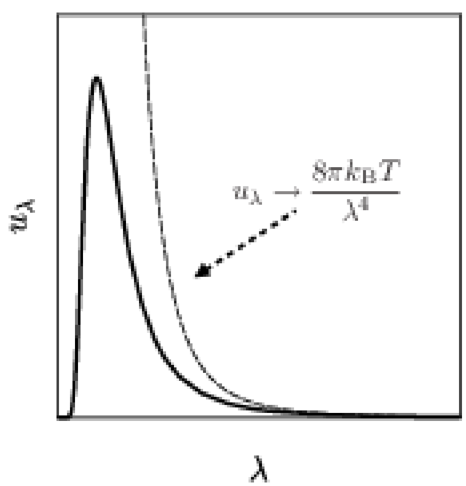
$$u_\nu \to \frac{8\pi k_B T \nu^2}{c^3} \qquad u_\lambda \to \frac{8\pi k_B T}{\lambda^4}$$

Rayleigh-Jeans law

Correct limit, before QM

Ultra-violet catastrophe from RJ





$$u = \int_0^\infty u_\lambda \,\mathrm{d}\lambda = \int_0^\infty \frac{8\pi k_B T \,\mathrm{d}\lambda}{\lambda^4} \to \infty$$

Why?

High frequency becomes harder and harder to excite, to a point they can't be excited.

This is a QM effect

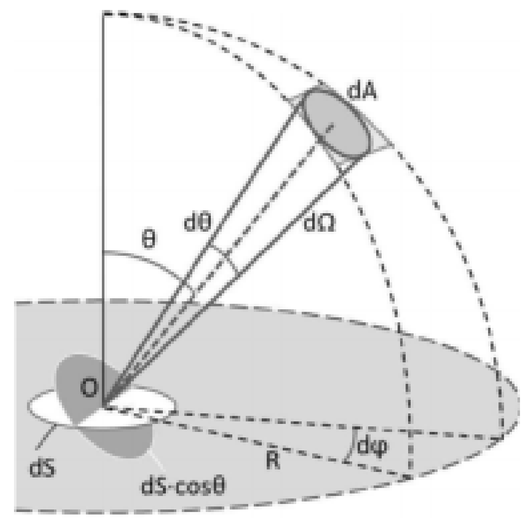
Radiance (surface brightness)

- B_ν: flux of radiation per steradian (sr) in a unit frequency interval.

  This is the power through an element of unit area, per unit frequency, **from** an element of solid angle. (units=$Wm^{-2}Hz^{-1}sr^{-1}$)

- We have 4π sr, thus: $B_\nu(T) = \frac{c}{4\pi}u_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{e^{\beta h\nu}-1}$

- Also: $B_\lambda(T) = \frac{c}{4\pi}u_\nu(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\beta hc/\lambda}-1}.$



https://math.stackexchange.com/questions/3121489/can-there-be-two-adjacent-solid-angles

Wien's law

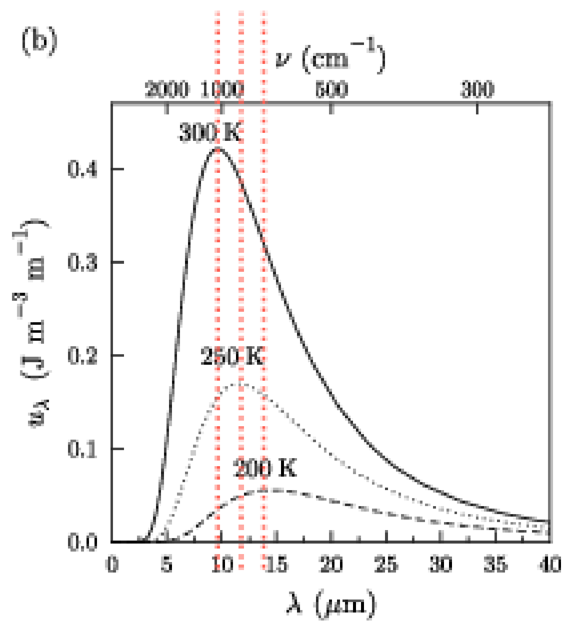
$\lambda_{\max} T = \text{a constant}$

$$u_\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{\beta hc/\lambda} - 1}$$

$$du_\lambda / d\lambda = 0$$

$$\Downarrow$$

$$\beta hc/\lambda_{\max} = \text{a constant.}$$



Example 1: At RT, objects will radiate at $10\mu m$ (infrared)

Example 2: At 2.7K, objects will radiate at 1mm (microwave)

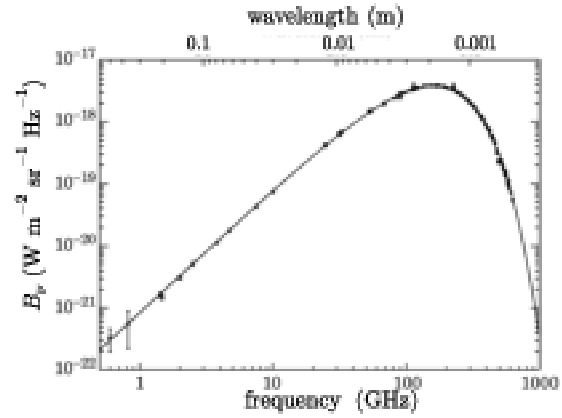
Cosmic microwave background radiation

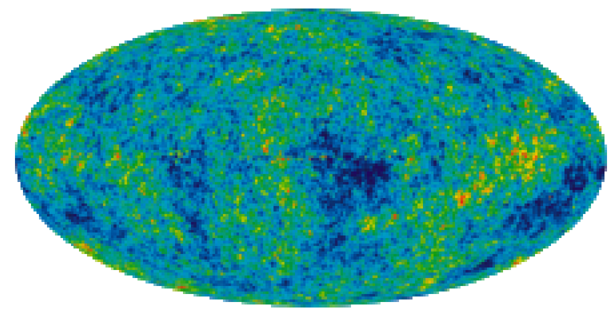
In 1978, Penzias and Wilson won the Nobel Prize for their observation (in 1963-1965) of a uniform microwave emission coming from **all directions** in the sky.

The spectral shape of this emission exhibits the distribution for black-body radiation of temperature **2.7K** with a peak in the emission spectrum at a wavelength of about 1 mm.

The radiation is **uniform and isotropic**!

This is one of the key pieces of evidence for the hot big bang model for the origin of the Universe. **It implies that there was a time when all of the Universe we see now was in thermal equilibrium.**





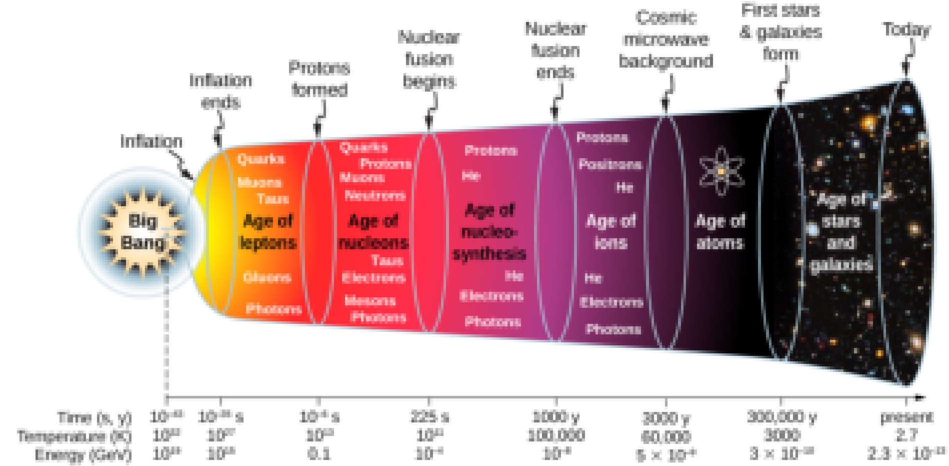
Wikipedia

*microwave are... mm long

More on the origin of the universe

We can make various inferences about the origin of the Universe from observations of the cosmic microwave background.

It can be shown that the energy density of radiation in the expanding Universe falls off as the fourth power of the scale factor.



openstax.org

From the Stefan-Boltzmann law, the energy density of radiation falls off as T^4, so temperature and scale factor are inversely proportional to one another, **so the Universe cools as it expands**.

When the Universe was much younger, it was much smaller and much hotter.

Extrapolating back in time, one finds that temperatures were such that physical conditions were very different.

The Einstein A and B coefficients

- If atoms are subjected to thermal radiation, the atoms can make transitions between different energy levels
- The atoms are in a **radiation field**, a "bath of photons". The RF has distribution u_ω
- What are the details of these transitions? We look at this treating atoms as two-level systems

$$u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

Spectral energy density

Two-level systems: (1) no field

Number of atoms in level 2:

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -A_{21}N_2$$

(the larger the number of excited atoms, the larger the change)

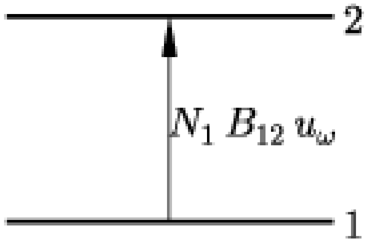
$$\Downarrow$$

$$N_2(t) = N_2(0)\mathrm{e}^{-t/\tau}$$

Radiative lifetime: $\tau \equiv 1/A_{21}$

Two-level systems: (2) with a field

An atom in level 1 can absorb a photon of energy $\hbar\omega$ to move to level 2



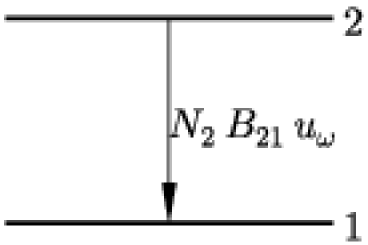
Absorption

Rate: $N_1 B_{12} u_\omega$

Increases with energy density and population of level 1

Levels are separated by $\hbar\omega$

(2) reverse effect is also possible

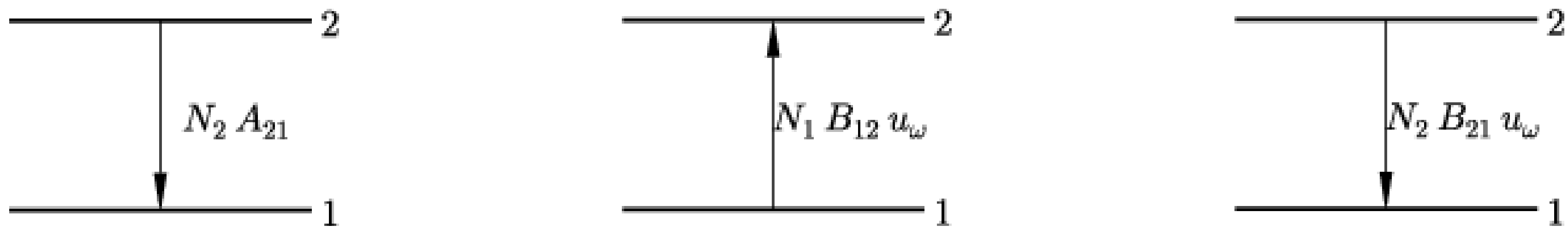


Stimulated emission

Rate: $N_2 B_{21} u_\omega$

This process involves two photons (absorbed and emitted)

We have three coefficients A_{21}, B_{12}, and B_{21}, "The Einstein A and B coefficients"



Spontaneous emission **Absorption** **Stimulated emission**

Steady state: $N_2 B_{21} u_\omega + N_2 A_{21} = N_1 B_{12} u_\omega$ (dynamic equilibrium)

Or: $$u_\omega = \frac{A_{21}/B_{21}}{(N_1 B_{12}/N_2 B_{21}) - 1}$$ **We also have** $$u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

At thermal equilibrium, we remember Boltzmann factor: $$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\beta\hbar\omega}$$ Including degeneracy g_1 and g_2

$$\Rightarrow \quad \frac{B_{21}}{B_{12}} = \frac{g_1}{g_2}$$

Gain

- The stimulated emission must be larger than the absorption rate for gain

$$N_2 B_{21} u_\omega > N_1 B_{12} u_\omega$$

$$\Downarrow$$

$$\frac{N_2}{g_2} > \frac{N_1}{g_1}$$

A **population inversion** is needed so that the number of atoms ("the population") in the upper state (per degenerate level) exceeds that in the lower state.

This is the principle of the **laser** (light amplification by stimulated emission of radiation).

However, in our two-level system such a population inversion is not possible in thermal equilibrium. For laser operation, it is necessary to have further energy levels to provide additional transitions: these can provide a mechanism to ensure that level 2 is **pumped** and that level 1 can **drain** away.

Summary

- Power emitted per unit area of a black-body surface at temperature T is given by σT^4 with

$$\sigma = \frac{\pi^2 k_B^4}{60 c^2 \hbar^3} = 5.67 \times 10^{-8}\,\mathrm{W m^{-2} K^{-4}}$$

- Radiation pressure due to black-body photons is equal to where is the energy density.
- Radiation pressure due to a collimated beam of light is equal to .
- The spectral energy density takes the form of a black-body distribution.
  - This form fits well to the experimentally measured form of the cosmic microwave background.
  - It is also important in the theory of lasers.