Supplement: exact differential

- In these units we will review some basics of exact differentials

Partial Differentials

$$x = x(y, z) \qquad \mathrm{d}x = \left(\frac{\partial x}{\partial y}\right)_z \mathrm{d}y + \left(\frac{\partial x}{\partial z}\right)_y \mathrm{d}z$$

$$z = z(x, y) \qquad \mathrm{d}z = \left(\frac{\partial z}{\partial x}\right)_y \mathrm{d}x + \left(\frac{\partial z}{\partial y}\right)_x \mathrm{d}y$$

$$\mathrm{d}x = \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial x}\right)_y \mathrm{d}x + \left[\left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x\right] \mathrm{d}y$$

$$\mathrm{d}x = \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial x}\right)_y \mathrm{d}x + \left[\left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x\right] \mathrm{d}y$$

$$\left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{\left(\frac{\partial z}{\partial x}\right)_y}$$

Reciprocal theorem

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

Reciprocity theorem

$$\left(\frac{\partial x}{\partial y}\right)_z = -\left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x$$

Exact differential

$$f(x,y) \longrightarrow \mathrm{d}f = \left(\frac{\partial f}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)\mathrm{d}y \longrightarrow \int_1^2 \mathrm{d}f = f(2) - f(1)$$

Path independent

Gradient of f: $F_1 = \left(\frac{\partial f}{\partial x}\right), \quad F_2 = \left(\frac{\partial f}{\partial y}\right) \longrightarrow F_1(x,y)\,\mathrm{d}x + F_2(x,y)\,\mathrm{d}y$

$$\int_1^2 F_1(x,y)\,\mathrm{d}x + F_2(x,y)\,\mathrm{d}y = \int_1^2 \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_1^2 \mathrm{d}f = f(2) - f(1)$$

Exact differential

$$\int_1^2 \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_1^2 \mathrm{d}f = f(2) - f(1)$$

→ Stokes Theorem

$$\nabla \times \boldsymbol{F} = 0$$

↓

$$\left(\frac{\partial F_2}{\partial x}\right) = \left(\frac{\partial F_1}{\partial y}\right)$$

↓

Condition:

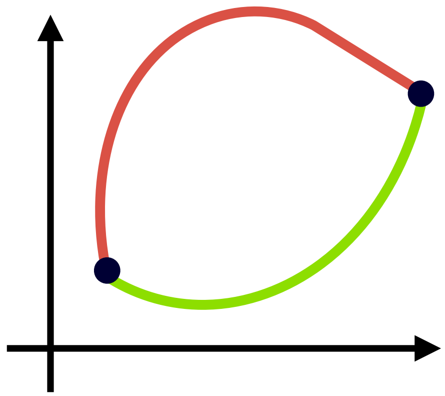
$$\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \left(\frac{\partial^2 f}{\partial y \partial x}\right)$$

Inexact differential $đ$

For an inexact differential this is not true and knowledge of the initial and final states is not sufficient to evaluate the integral: you have to know which path was taken.

$$\int_1^2 \mathrm{d}f = f(2) - f(1) \longrightarrow \oint \mathrm{d}f = 0$$

$$\int_1^2 đf \neq f(2) - f(1) \longrightarrow \oint đf \neq 0$$

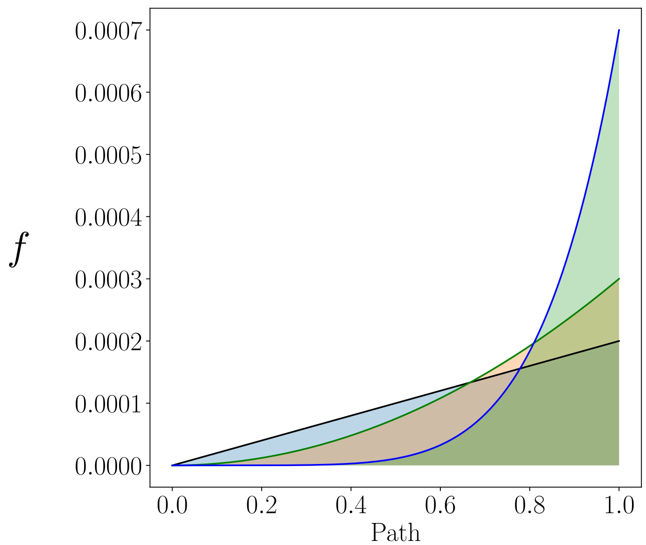


For thermal physics, a crucial point to remember is that functions of state have exact differentials.

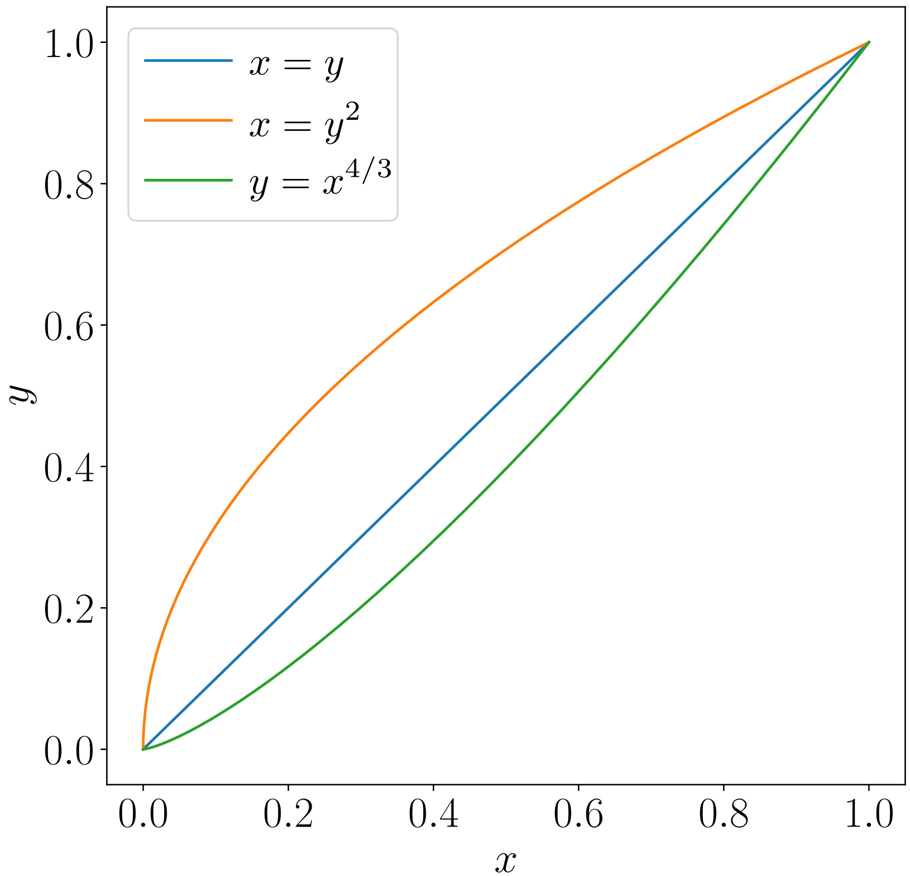
Example

Problem: integrate from (0,0) to (1,1)

$$\mathrm{d}f = y\mathrm{d}x + x\mathrm{d}y$$ Exact



Three possible paths

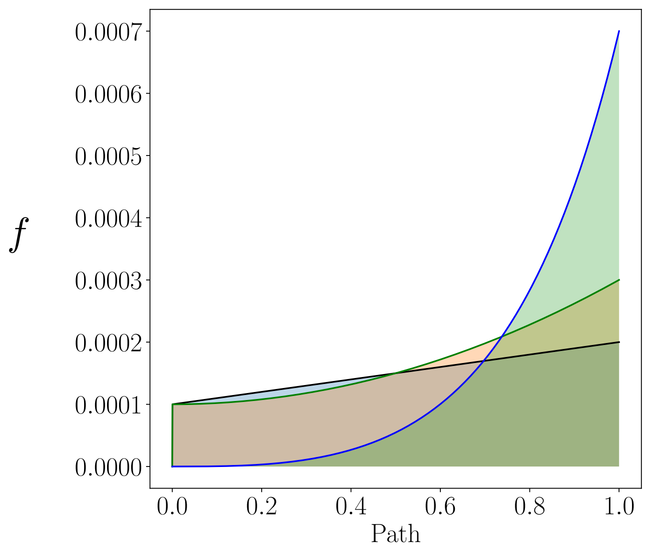


Same surface areas *under the curves*!

Example

Problem: integrate from (0,0) to (1,1)

$$\mathrm{d}f = y\mathrm{d}x + x^2\mathrm{d}y$$ Inexact



Different surface areas!

Three possible paths